On geometric inequalities related to fractional integration

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Declaration

I hereby declare that this thesis sumbitted was composed by myself for the degree of Doctor of Philosophy in Mathematics under the guidance and supervision of Professor Carbery at University of Edinburgh. Where other sources of information have been used, they have been explicitly stated and acknowledged in this thesis.

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Dedicated to my parents

Lay summary

A well known geometric extremal problem, isodiametric problem, that is, amongst all sets with a given diameter, which has the largest volume? The answer is balls. Using symmetrisation, known as symmetric decreasing rearrangement, we find the diameter of a set will decrease if it is replaced by its spherical rearrangement. Symmetric decreasing rearrangements manipulate measurable sets while preserving their volumes, so they are extremely useful analytic tools to deal with many geometric extremal problems. For instance, Rearrangement gives the isoperimetric inequality that amongst all bodies with a given volume, the ball has the smallest surface area. It also gives the Brunn-Minkowski inequality that the volume of the sum of two sets decreases if replacing the two sets by their spherical rearrangements.

One objective of this thesis is to study some geometric problems related to the isodiametric problem in more general settings. For example, it is well known that given a compact convex set, there exists a simplex of maximal volume contained in it. The question is, amongst all sets of given volume, which has simplices of least maximal volume with vertices in it? It is solved by using the method of another symmetrisation rearrangement, known as Steiner symmetrisation procedure. Along this direction we will focus on the extremal sets of its multilinear perspective generalizations. Another analogue of isodiametric problems is to replace the Euclidean space by the space of real matrices, and the Euclidean norm by the determinant of matrices. To deal with these problems rearrangements play an important role.

In this thesis we also establish some functional versions of inequalities derived from isodiametric problems above. Particular emphasis will be given to the sharp constants and optimisers of these geometric functional inequalities. The determination of the optimisers requires rearrangements as well, similar to the sharp versions of some famous integral inequalities: the Hardy-Littlewood-Sobolev inequality, Young's inequality, the Sobolev inequality and so on. For example, the optimisers of the Hardy-Littlewood-Sobolev inequality and Young's inequality are determined by maximizing over spherically symmetric functions by applying the Riesz rearrangement inequality. Similarly, applying the Pólya-Szegő inequality, the optimisers of the Sobolev inequality is determined by minimizing over spherically symmetric functions.

Abstract

The first part of this thesis establishes a series of geometric ineqalities related to fractional integration in some geometric settings, including bilinear and multilinear forms. In the second part of this thesis, we study some kinds of rearrangement inequalities. In particular, some applications of rearrangement inequalities will be given, for instance, the determination of the extremals of some geometric problems. By competing symmetries and rearrangement inequalities, we prove the sharp versions of geometric inequalities introduced in the first part in Euclidean spaces. Meanwhile, there are the corresponding conformally equivalent formulations in unit sphere and in hyperbolic space. The last part is about collaborative work on the regularity of the Hardy-Littlewood maximal functions. We give a simple proof to improve Tanaka's result of the paper entitled "A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function". Our proof is based on the behaviour of the local maximum of the non-centered Hardy-Littlewood maximal function.

Chapter 1

Introduction

1.1 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $n \geq 1$, and $|\cdot|$ denotes the Lebesgue measure on Euclidean space $\mathbb{R}^n$ and the norm in a Hilbert space. The notation $A \lesssim B$ means there exists a positive constant C independent of the essential variables such that $A \leq CB$. The notation $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Throughout this thesis, all functions considered are nonnegative.

The Hardy-Littlewood-Sobolev inequality [44] states that

$$\| \int_{\mathbb{R}} f(x) \frac{1}{|x-y|^{2-\frac{1}{p}-\frac{1}{q}}} dx \|_{q'} \leq C_{p,q} \, \|f\|_p$$

holds for $1 < p,\ q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$ and all functions $f \in L^p(\mathbb{R})$. Applying Hölder's inequality gives the forward Hardy-Littlewood-Sobolev inequality (1.1.1):

$$| \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(y) \frac{1}{|x-y|^{2-\frac{1}{p}-\frac{1}{q}}} dxdy | \leq C_{p,q} \, \|f\|_p \|g\|_q \tag{1.1.1}$$

holds for all functions $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, $1 < p,\ q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$.

It is already known that the forward Hardy-Littlewood-Sobolev inequality (1.1.1) and the inverse Hardy-Littlewood-Sobolev inequality (1.1.2) which follows are equivalent (see [44]). For $0 < p,\ q < 1$, and all functions $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, we have

$$\|f\|_p \|g\|_q \leq C_{p,q} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)g(y)| |x-y|^{\frac{1}{p}+\frac{1}{q}-2} dxdy. \tag{1.1.2}$$

The proof of their equivalence is as follows. First look at (1.1.1) $\Rightarrow$ (1.1.2), mainly using Hölder's Inquality.

Choose any σ such that $\sigma > 1$, and denote $\gamma = \frac{1}{p} + \frac{1}{q} - 2$. Then

$$\begin{aligned}
&\|f\|_p^p \ \|g\|_q^q \\
&= \int\int f(x)^p g(y)^q ds\ dt \\
&= \int\int f(x)^{\frac{1}{\sigma}} g(y)^{\frac{1}{\sigma}} |x-y|^{\frac{\gamma}{\sigma}} f(x)^{p-\frac{1}{\sigma}} g(y)^{q-\frac{1}{\sigma}} |x-y|^{-\frac{\gamma}{\sigma}} dxdy \\
&\leq (\int\int (f(x)^{\frac{1}{\sigma}} g(y)^{\frac{1}{\sigma}} |x-y|^{\frac{\gamma}{\sigma}})^{\sigma} dsdt)^{\frac{1}{\sigma}} (\int\int (f(x)^{p-\frac{1}{\sigma}} g(y)^{q-\frac{1}{\sigma}} |x-y|^{-\frac{\gamma}{\sigma}})^{\sigma'} dsdt)^{\frac{1}{\sigma'}} \\
&= (\int\int f(x)g(y)|x-y|^{\gamma} dxdy)^{\frac{1}{\sigma}} (\int\int f(x)^{(p-\frac{1}{\sigma})\sigma'} g(y)^{(q-\frac{1}{\sigma})\sigma'} |x-y|^{-\frac{\gamma\sigma'}{\sigma}} dxdy)^{\frac{1}{\sigma'}} \\
&\lesssim (\int\int f(x)g(y)|x-y|^{\gamma} dxdy)^{\frac{1}{\sigma}} \ (\|f^{(p-\frac{1}{\sigma})\sigma'}\|_{p_1} \|g^{(q-\frac{1}{\sigma})\sigma'}\|_{q_1})^{\frac{1}{\sigma'}},
\end{aligned}$$

where $p_1 = \frac{p}{(p-\frac{1}{\sigma})\sigma'} > 1$, $q_1 = \frac{q}{(q-\frac{1}{\sigma})\sigma'} > 1$.
For the last inequality, the reason why

$$\int\int f(x)^{(p-\frac{1}{\sigma})\sigma'} g(y)^{(q-\frac{1}{\sigma})\sigma'} |s-t|^{-\frac{\gamma\sigma'}{\sigma}} dxdy \lesssim \|f^{(p-\frac{1}{\sigma})\sigma'}\|_{p_1} \|g^{(q-\frac{1}{\sigma})\sigma'}\|_{q_1}$$

is because $p_1 > 1$, $q_1 > 1$, and they meet $\frac{\gamma\sigma'}{\sigma} = 2 - \frac{1}{p_1} - \frac{1}{q_1}$.
Then we obtain

$$\|f\|_p^p \ \|g\|_q^q \lesssim (\int\int f(x)g(y)|x-y|^{\gamma} dxdy)^{\frac{1}{\sigma}} \ \|f\|_p^{p-\frac{1}{\sigma}} \|g\|_q^{q-\frac{1}{\sigma}},$$

which gives

$$\|f\|_p \ \|g\|_q \leq C_{p,q} \int\int \ f(x)g(y)|x-y|^{\frac{1}{p}+\frac{1}{q}-2} dxdy.$$

Conversely, suppose for $0 < p, q < 1$,

$$\|f\|_p \ \|g\|_q \leq C_{p,q} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)g(y)||x-y|^{\frac{1}{p}+\frac{1}{q}-2} dxdy.$$

Let $f = g = \chi_B$ where B is a measurable set in $\mathbb{R}$, then

$$|B|^{1/p}|B|^{1/q} \lesssim \int_B\int_B |x-y|^{\frac{1}{p}+\frac{1}{q}-2} dxdy. \tag{1.1.3}$$

Considering restricted weak-type estimates, let

$$T(f,g) = \int_{\mathbb{R}}\int_{\mathbb{R}} \ f(x)g(y)|x-y|^{-\gamma} dxdy,$$

where $\gamma = 2 - \frac{1}{p} - \frac{1}{q}$. For any measurable set $E, F \in \mathbb{R}$,

$$T(\chi_E, \chi_F) = \int\int \chi_E(x)\chi_F(y)|x-y|^{-\gamma} dxdy.$$

First integrate for y, that is, for any $x \in E$,

$$\begin{aligned}\int_F |x-y|^{-\gamma} dy &= \sum_{j=-\infty}^{\infty} \int_F |x-y|^{-\gamma} \chi_{|x-y|\sim 2^j}(y) dy \\ &= \sum_{2^j<|F|} \int_F |x-y|^{-\gamma} \chi_{|x-y|\sim 2^j}(y) dy + \sum_{2^j\geq|F|} \int_F |x-y|^{-\gamma} \chi_{|x-y|\sim 2^j}(y) dy \\ &\lesssim \sum_{2^j<|F|} 2^{-j\gamma} 2^j + \sum_{2^j\geq|F|} 2^{-j\gamma}|F|,\end{aligned}$$

where the last inequality holds because in $\mathbb{R}$ we have

$$|\{y \in F : |x-y| \sim 2^j\}| \leq |B(x,2^j) \cap F| \lesssim \min\{2^j,\ |F|\}. \tag{1.1.4}$$

Therefore,

$$\int_F |x-y|^{-\gamma} dt \lesssim \sum_{2^j<|F|} 2^{-j\gamma} 2^j + \sum_{2^j\geq|F|} 2^{-j\gamma}|F| \lesssim |F|^{1-\gamma}.$$

That implies

$$T(\chi_E,\chi_F) = \int\int \chi_E(x)\chi_F(y)|x-y|^{-\gamma} dxdy \lesssim |E||F|^{1-\gamma}.$$

By symmetry, there are also

$$T(\chi_E,\chi_F) = \int\int \chi_E(x)\chi_F(y)|x-y|^{-\gamma} dxdy \lesssim |E|^{1-\gamma}|F|.$$

Applying restricted weak-type interpolation [44] gets

$$T(f,g) = \int\int f(x)g(y)|x-y|^{-\gamma} dxdy \lesssim \|f\|_p\|g\|_q,$$

for $1-\gamma < \frac{1}{p}, \frac{1}{q} < 1$, that is $1 < p,q < \infty$.

Although we do not need (1.1.2) and (1.1.3) to deduce (1.1.4) in the current setting, in more general setting inputs such as (1.1.2) and (1.1.3) are required to close the argument.

As a result, from the inverse Hardy-Littlewood-Sobolev inequality (1.1.2) we deduce that for $0 < p,\ q < r < \infty$ and all measurable functions f, g,

$$\|f^r\|_{\frac{p}{r}}\|g^r\|_{\frac{q}{r}} \leq C_{\frac{p}{r},\frac{q}{r}} \int_{\mathbb{R}}\int_{\mathbb{R}} |f^r(x)g^r(y)||x-y|^{\frac{r}{p}+\frac{r}{q}-2} dxdy.$$

Then, for $0 < p,\ q < r < \infty$ and all functions $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, we have

$$\|f\|_p\ \|g\|_q \leq C^{\frac{1}{r}}_{\frac{p}{r},\frac{q}{r}}\ \|f(x)g(y)|x-y|^{\frac{1}{p}+\frac{1}{q}-\frac{2}{r}}\|_{L^r(dxdy)}. \tag{1.1.5}$$

It is natural to ask what the inequality (1.1.5) leads to if we let $r \to \infty$, and what happens to the constant $C^{\frac{1}{r}}_{\frac{p}{r},\frac{q}{r}}$ as $r \to \infty$. One approach is to consider the behaviour of $C^{\frac{1}{r}}_{\frac{p}{r},\frac{q}{r}}$ as $r \to \infty$, but as we are interested in geometrical questions, we prefer a more direct approach. The natural conjecture is that for any $0 < p, q < \infty$ and all functions $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$

$$\|f\|_p \ \|g\|_q \leq C_{p,q} \sup_{x,y} |f(x)g(y)||x-y|^\gamma, \tag{1.1.6}$$

where the supremum sup is the essential supremum of function throughout the thesis. Clearly, we take $\gamma = \frac{1}{p} + \frac{1}{q}$ which follows from homogeneity, as we now show.

Suppose that $\|f\|_p \ \|g\|_q \leq C_{p,q} \sup_{x,y} |f(x)g(y)||x-y|^\gamma$ holds. Then consider functions $f(\frac{\cdot}{R})$, $g(\frac{\cdot}{R})$ for all $R > 0$:

$$\|f(\frac{\cdot}{R})\|_p = R^{\frac{1}{p}}\|f\|_p, \ \|g(\frac{\cdot}{R})\|_q = R^{\frac{1}{q}}\|g\|_p,$$

and

$$\begin{aligned}\sup_{x,y} |f(\frac{x}{R})g(\frac{y}{R})||x-y|^\gamma &= R^\gamma \sup_{s,t} |f(\frac{x}{R})g(\frac{y}{R})||\frac{x}{R} - \frac{y}{R}|^\gamma \\ &= R^\gamma \sup_{x,y} |f(x)g(y)||x-y|^\gamma.\end{aligned}$$

From (1.1.6),

$$R^{\frac{1}{p}+\frac{1}{q}}\|f\|_p \ \|g\|_p \leq C_{p,q} \ R^\gamma \sup_{x,y} |f(s)g(t)||x-y|^\gamma.$$

This indicates for all $R > 0$,

$$R^{\frac{1}{p}+\frac{1}{q}} \leq C_{p,q} \ R^\gamma,$$

which implies $\gamma = \frac{1}{p} + \frac{1}{q}$.

If we consider the simple case when f, g are supported in an interval $E \subset \mathbb{R}$, we find

$$\|f\|_p \ \|g\|_q \leq \|f\|_\infty \ \|g\|_\infty |E|^{\frac{1}{p}+\frac{1}{q}} = \sup_x |f(x)| \ \sup_y |g(y)| \ \sup_{x,y} |x-y|^{\frac{1}{p}+\frac{1}{q}},$$

where $|E|$ is the Lebesgue measure of E. Clearly the right side of (1.1.6) is in principle smaller than $\sup_x |f(x)| \ \sup_y |g(y)| \ \sup_{x,y} |x-y|^{\frac{1}{p}+\frac{1}{q}}$.

1.2 Some Geometric Inequalities Related to Fractional Integration

In Chapter 2 Section 2.1, we shall give a positive answer for the bilinear type of geometric inequality (1.1.6) as follows. For any $0 < p, q < \infty$ and all functions

$f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $\gamma = \frac{n}{p} + \frac{n}{q}$ (homogeneity condition),

$$\|f\|_{L^p(\mathbb{R}^n)} \; \|g\|_{L^q(\mathbb{R}^n)} \leq C_{p,q,n} \sup_{x,y} |f(x)g(y)||x-y|^\gamma. \tag{1.2.1}$$

Also we investigate the analogues of inequality (1.1.6) in more general settings, such as in a metric space with a certain geometric measure. The question is, for any metric space $(M,\ d)$ with a σ-finite, nonnegative Borel measure μ on M, does there exist a finite constant $C_{p,q,\gamma}$ independent of functions f, g such that

$$\|f\|_{L^p(d\mu)} \; \|g\|_{L^q(d\mu)} \leq C_{p,q,\gamma} \sup_{x,y} |f(x)g(y)| d(x,y)^\gamma \tag{1.2.2}$$

holds for any $0 < p, q < \infty$ and all functions $f \in L^p(d\mu), g \in L^q(d\mu)$?

It is also natural to conjecture that the following two multilinear versions of inequality (1.1.5) hold on Euclidean spaces.

(i) the multilinear determinant form:

For any $0 < p_j < \infty$ and all functions $f_j \in L^{p_j}(\mathbb{R}^n)$, $1 \leq j \leq n+1$, $\gamma = \displaystyle\sum_{j=1}^{n+1} \frac{1}{p_j}$,

$$\prod_{j=1}^{n+1} \|f_j\|_{p_j} \leq C \sup_{y_j} \prod_{j=1}^{n+1} |f_j(y_j)| \det(y_1, \ldots, y_{n+1})^\gamma. \tag{1.2.3}$$

The notation $\det(y_1, \ldots, y_{n+1})$ denotes $n!$ times the Euclidean n-dimensional volume of the simplex with vertices $y_1, \ldots, y_{n+1}$, so $\det(y_1, \ldots, y_{n+1}) \geq 0$ throughout the thesis. Similar to (1.2.1), $\gamma = \displaystyle\sum_{j=1}^{n+1} \frac{1}{p_j}$ also follows from homogeneity.

In Chapter 2 Section 2.2, we will prove the multilinear determinant form inequality (1.2.3) is true. Moreover, combining with Gressman's work [21] we investigate what inequality (1.2.3) would be like in more general settings apart from in the Euclidean space cases , for instance, in a real finite-dimensional Hilbert space H with a certain geometric measure as discussed in [21].

The second possible multilinear form we study is to replace the determinant form by "product form" as follows.

(ii) the multilinear product form:

Let $r_{ij} > 0$ and $r_{ij} = r_{ji}$. For any $0 < p_j < \infty$ and all functions $f_j \in L^{p_j}(\mathbb{R}^n)$, $j = 1, \ldots, N$, $\displaystyle\sum_{j=1}^{N} \frac{1}{p_j} = \frac{1}{n} \sum_{1 \leq i < j \leq N} r_{ij}$, and $N \in \mathbb{N}$,

$$\prod_{j=1}^{N} \|f_j\|_{p_j} \leq C_{p_j, r_{ij}, n, N} \sup_{y_j} \prod_{j=1}^{N} |f_j(y_j)| \prod_{1 \leq i < j \leq N} |y_i - y_j|^{r_{ij}}. \tag{1.2.4}$$

The condition $\displaystyle\sum_{j=1}^{N} \frac{1}{p_j} = \frac{1}{n} \sum_{1 \leq i < j \leq N} r_{ij}$ follows from homogeneity.

The results related to the product type geometric inequaltiy are described in Chapter 2 Subsection 2.2.3. We give the necessary and sufficeint conditions for the trilinear case of (1.2.4) to hold. Combining with Beckner's work [3], we prove the sufficient conditions for (1.2.4) to hold for other general cases.

After establishing these bilinear and multilinear geometric inequalities introduced above, it is natural to find their optimal constants and extremal functions. In Chapter 4, we will investigate and prove the sharp versions of bilinear form and multilinear form in the Euclidean spaces settings with L^p bounds. Meanwhile, there are the corresponding conformally equivalent formulations in unit sphere $\mathbb{S}^n$ and in hyperbolic space $\mathbb{H}^n$.

1.3 Some Matrix Inequalities with a Geometrical Flavour

The second goal of this thesis is to find the optimisers of the bilinear and the multilinear determinant geometric inequalities introduced above for characteristic functions. Particular emphasis will be given to a new type of matrix inequality which has a geometrical flavour. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $n \geq 1$, and $|\cdot|$ denotes the Lebesgue measure on Euclidean space $\mathbb{R}^n$ and the absolute value on $\mathbb{R}$. Denote $\mathfrak{M}^{n\times n}$ a set of all $n \times n$ real matrices. Let $B(0,r)$ be the ball centred at 0 with radius r. For $A \subset \mathbb{R}^n$ of finite Lebesgue measure, we define the symmetric rearrangement of set A as

$$A^* := \{x : |x| < r\} \equiv B(0,r), \text{ with } |A^*| = |A|.$$

That is, $v_n r^n = |A|$, where v_n is the volume of unit ball in $\mathbb{R}^n$. Let f be a measurable function that vanishes at infinity, in the sense that all its positive level sets have finite measure,

$$|\{x : |f(x)| > t\}| < \infty, \text{ for all } t > 0.$$

We then define the symmetric decreasing rearrangement of nonnegative measurable function f as

$$f^*(x) := \int_0^\infty \chi_{\{f>t\}^*}(x)dt,$$

where $\chi_{\{f>t\}}$ is the characteristic function of the level set $\{x : f(x) > t\}$.

One can easily check that for any measurable set $E \subset \mathbb{R}^n$

$$\sup_{x\in E^*} |x| \leq \sup_{x\in E} |x|. \tag{1.3.1}$$

Then we have the following rearrangement inequality

$$\sup_{x,y\in E^*} |x-y| \leq \sup_{x,y\in E} |x-y|. \tag{1.3.2}$$

One way to obtain this is as follows,

$$\sup_{x,y\in E} |x-y| = \sup_{z\in E-E} |z| \geq \sup_{z\in (E-E)^*} |z|. \tag{1.3.3}$$

For any $A, B \in \mathbb{R}^n$ of finite Lebesgue measure, it follows from the Brunn-Minkowski inequality that

$$A^* + B^* \subset (A+B)^*. \tag{1.3.4}$$

Applying (1.3.4) in (1.3.3) implies

$$\sup_{x,y\in E} |x-y| \geq \sup_{z\in(E-E)^*} |z| \geq \sup_{x\in E^*, y\in E^*} |x-y|,$$

which gives (1.3.2).

Let E be a measurable set of finite volume in $\mathbb{R}^n$. By the definition of the symmetric rearrangement,

$$E^* = B(0,r), \text{ with } v_n r^n = |E|.$$

Clearly,

$$\sup_{x\in E^*} |x| = r, \ \sup_{x,y\in E^*} |x-y| = 2r.$$

Based on (1.3.1) and (1.3.2) we have the following sharp bilinear inequalities

$$|E| \leq v_n \sup_{x\in E} |x|^n, \tag{1.3.5}$$

$$|E| \leq \frac{v_n}{2^n} \sup_{x,y\in E} |x-y|^n. \tag{1.3.6}$$

Moreover, both optimisers of (1.3.5) and (1.3.6) are balls in $\mathbb{R}^n$. Inequality (1.3.6) is an isodiametric inequality, that is, amongst all sets with given diameter the ball has maximal volume.

We now go on to study the analogues of (1.3.5) and (1.3.6) where we replace the distance norm by a volume or determinant, so the question becomes that of studying inequalities of the form

$$|E| \leq A_n \sup_{\substack{y_j\in E\\ j=1,\dots,n}} \det(0, y_1, \dots, y_n), \tag{1.3.7}$$

and

$$|E| \leq B_n \sup_{\substack{y_j\in E\\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}), \tag{1.3.8}$$

which are supposed to hold for any measurable set E in $\mathbb{R}^n$. As before,

$$\det(y_1, \dots, y_{n+1}) := n!\mathrm{vol}(\mathrm{co}\{y_1, \dots, y_{n+1}\}).$$

The precise value of $\det(y_1, \dots, y_{n+1})$ is the absolute value of the determinant of the matrix $(y_1 - y_{n+1}, \dots, y_n - y_{n+1})_{n\times n}$. In the special case when $n = 1$, they become of the type (1.3.5) and (1.3.6) automatically. Note that both (1.3.7) and (1.3.8) are $\mathrm{GL}_n(\mathbb{R})$ invariant, and (1.3.8) is translation invariant while (1.3.7) is not. Actually, it is enough to study convex measurable sets in $\mathbb{R}^n$, since

$$\sup_{\substack{y_j \in E\\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) = \sup_{\substack{y_j \in \mathrm{co}(E)\\ j=1,\dots,n}} \det(0, y_1, \dots, y_n),$$

and

$$\sup_{\substack{y_j \in E\\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) = \sup_{\substack{y_j \in \mathrm{co}(E)\\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}).$$

We are interested in the best constants A_n, B_n and their optimsers. It is not hard to deduce that the best constant A_n and B_n are related by

$$B_n \le A_n \le (n+1)B_n. \tag{1.3.9}$$

Indeed, the translation invariance of (1.3.8) allows us to assume that $0 \in E$. Then $B_n \le A_n$ follows immediately. On the other hand, by the basic determinant property we have

$$\det(y_1, \dots, y_{n+1}) \le \sum_{j=1}^{n+1} \det(0, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n),$$

which implies that

$$\sup_{\substack{y_j \in E\\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) \le (n+1) \sup_{\substack{y_j \in E\\ j=1,\dots,n}} \det(0, y_1, \dots, y_n).$$

That completes $A_n \le (n+1)B_n$. So in the special case when $n = 1$, we have $A_1 = 2$, $B_1 = 1$ that follows from (1.3.5) and (1.3.6).

Geometrically, the right side of (1.3.8) relates to the maximal volume of n-simplex whose vertices are in E. The relationship between the maximal volume of the n-simplex whose vertices are in E and the measure of E has been studied before (see [24], [38]). It is well known that by compactness given a compact convex set $E \subset \mathbb{R}^n$, there exists a simplex $T \subset E$ of maximal volume. Let F be a facet of T, v the opposite vertex, and H the hyperplane through v parallel to F. Then H supports E, since otherwise one would obtain a contradiction to the maximality of the volume of T. Since F is an arbitrary facet of T, T is contained in the simplex $-n(T - c) + c$, where c is the centroid of T. See [24] for details. So $T \subset E \subset -n(T - c) + c$, and thus

$$|E| \le n^n |T|. \tag{1.3.10}$$

which implies that

$$B_n \le n^n, \;\; A_n \le (n+1)n^n.$$

In 1950, Macbeath [38] already gave the sharp version of (1.3.10) and (1.3.8) as follows. Given a compact convex set $E \subset \mathbb{R}^n$, denote $\mathfrak{B}_m$ the set of convex polytopes with at most m vertices in E, and denote $\mathfrak{C}_m$ the set of convex polytopes with at most m vertices in E^*. Then

$$\sup_{T' \in \mathfrak{C}_m} |T'| \leq \sup_{T \in \mathfrak{B}_m} |T|. \tag{1.3.11}$$

So when $m = n + 1$, (1.3.11) gives

$$\sup_{\substack{y_j \in E^* \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) \leq \sup_{\substack{y_j \in E \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}).$$

Moreover the problem is clearly affine invariant, thus the extremising sets turn out to be balls and ellipsoids for (1.3.8). Because the maximal simplex with vertices on a ball is the regular simplex with all sides equal, we can obtain the corresponding best constant B_n. However, we do not believe that the sharp value of A_n in (1.3.7) has been given previously.

In Chapter 3 Section 3.3 we shall give an alternative method to derive (1.3.7) and (1.3.8) with sharp constants A_n, B_n and also conclude Macbeath's work (1.3.11) when $m = n + 1$. More generally, we obtain the extremal sets of the following inequalities,

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n}} \leq A_n \sup_{y_1 \in E_1, \dots, y_n \in E_n} \det(0, y_1, \dots, y_n), \tag{1.3.12}$$

and

$$\prod_{j=1}^{n+1} |E_j|^{\frac{1}{n+1}} \leq B_n \sup_{y_1 \in E_1, \dots, y_{n+1} \in E_{n+1}} \det(y_1, \dots, y_{n+1}), \tag{1.3.13}$$

where $E_j \subset \mathbb{R}^n$. However, the sharp constant B_n of (1.3.13) has not been given in this thesis. Before studying (1.3.12)-(1.3.13), in Section 3.3 we introduce some rearrangement inequalities which together with some work in [12] establish that (1.3.7)-(1.3.8) and (1.3.12)-(1.3.13) are extremised by balls centred at 0. A key ingredient will be Lemma 4.7 of [12], stating that for any $E_j \subset \mathbb{R}$ of finite Lebesgue measure, and $a_j \in \mathbb{R}$, $j = 1, \dots, l$,

$$\sup_{x_j \in E_j^*} |\sum_{j=1}^{l} a_j x_j| \leq \sup_{x_j \in E_j} |\sum_{j=1}^{l} a_j x_j|. \tag{1.3.14}$$

See Lemma 3.2.4 for the proof.

There is another class of inequalities concerning analogues of (1.3.5), (1.3.6) where we replace the underlying Euclidean space $\mathbb{R}^n$ by the space of $n \times n$ real matrices, and the Euclidean norm by $|\det(A)|$. For example, Christ first studied this type of inequality in [15]. Here "det" becomes ordinary determinant of a matrix.

Sublemma 14.1. [15] *For any $n \geq 1$ there exists $C \in \mathbb{R}^+$ with the following property. Let $E \subset \mathfrak{M}^{n \times n}$ be a compact convex set satisfying $|E| < \infty$ and $E =$*

$-E$. *Then there exists* $A \in E$ *satisfying*

$$|\det(A)| \geq C|E|^{\frac{1}{n}}, \tag{1.3.15}$$

where $|\cdot|$ *denotes the Lebesgue measure on Euclidean space* $\mathbb{R}^{n^2}$ *and the absolute value on* $\mathbb{R}$.

Lemma 13.2. [15] *For any* $n \geq 1$ *there exists* $c, C \in \mathbb{R}^+$ *and* $k \in \mathbb{N}$ *with the following property. Let* E *be a measurable set in* $\mathfrak{M}^{n\times n}$ *satisfying* $|E| < \infty$. *Then there exist* $T_1, \ldots, T_k \in E$ *and coefficients* $s_j \in \mathbb{Z}$ *satisfying* $|s_j| \leq c$, $\sum\limits_{j=1}^{k} s_j = 0$, *such that*

$$|\det(\sum_{j=1}^{k} s_j T_j)| \geq C|E|^{\frac{1}{n}}. \tag{1.3.16}$$

Remarks 1.

1. Let $\widetilde{E} = E - A := \{T - A : T \in E\}$ with $A \in \mathfrak{M}^{n\times n}$, then by Lemma 13.2 there exist $T_1, \ldots, T_k \in E$ and $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, $\sum\limits_{j=1}^{k} s_j = 0$, such that

$$|\det(\sum_{j=1}^{k} s_j(T_j - A))| = |\det(\sum_{j=1}^{k} s_j T_j)| \geq C|E|^{\frac{1}{n}} = C|\widetilde{E}|^{\frac{1}{n}}, \tag{1.3.17}$$

which shows (1.3.16) has a translation invariant property that (1.3.15) lacks.

2. Based on the translation variant property, we have an equivalent form of Lemma 13.2: there exists $c, C \in \mathbb{R}^+$ such that for any $E \subset \mathfrak{M}^{n\times n}$ we can always select $T_1, \ldots, T_k \in E$ and coefficients $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, such that $|\det(\sum\limits_{j=1}^{k} s_j T_j)| \geq C|E|^{\frac{1}{n}}$.

The equivalence is as follows. Supposing $A \in E$, denote $\widetilde{E} = E - A$. Then if there exist $\overline{T}_1 = T_1 - A, \ldots, \overline{T}_k = T_k - A \in \widetilde{E}$, where $T_j \in E$, $1 \leq j \leq k$, and there exist $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, such that

$$|\det(\sum_{j=1}^{k} s_j \overline{T}_j)| \geq C|\widetilde{E}|^{\frac{1}{n}}.$$

That is,

$$|\det(s_1 T_1 + \cdots + s_k T_k - (s_1 + \cdots + s_k)A)| \geq C|\widetilde{E}|^{\frac{1}{n}} = C|E|^{\frac{1}{n}},$$

which satisfies the conditions of Lemma 13.2.

More specifically, when proving Lemma 13.2 Christ [15] gave that under the same hypothesis of Lemma 13.2, there exist $A_j \in E$ and $s_j \in \{0, 1\}$, $j = 1, \ldots, n$, such that

$$|\det(\sum_{j=1}^{n} s_j A_j)| \geq C|E|^{\frac{1}{n}}, \tag{1.3.18}$$

which implies that for any measurable $E \subset \mathfrak{M}^{n\times n}$

$$\sup_{\substack{A_1,\dots,A_n\in E\\ s_1,\dots,s_n\in\{0,1\}}} |\det(s_1A_1+\cdots+s_nA_n)| \gtrsim_n |E|^{\frac{1}{n}}. \tag{1.3.19}$$

In Chapter 3 we will improve Sublemma 14.1 and Lemma 13.2 [15] as follows, mainly relying on the rearrangement inequality (1.3.14).

Main Theorem. *There exists a finite constant $\mathcal{C}_n$ such that for any measurable sets $E_j \subset \mathfrak{M}^{n\times n}$ of finite measure, $j = 1,\dots,n$,*

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n^2}} \le \mathcal{C}_n \sup_{\substack{A_j\in E_j\\ j=1,\dots,n}} |\det(A_1+\cdots+A_n)|. \tag{1.3.20}$$

The main theorem implies (1.3.15) holds for all compact convex sets in $\mathfrak{M}^{n\times n}$ and extends Lemma 13.2 as described below. In particular, we see from the Main Theorem that all the s_j in (1.3.20) can be taken to be 1.

Corollary A. *There exists a finite constant $\mathcal{A}_n$ such that for any measurable set $E \subset \mathfrak{M}^{n\times n}$ of finite measure, for any non-zero scalar $\lambda_j \in \mathbb{R}$, $j = 1,\dots,n$,*

$$(\prod_{j=1}^{n} |\lambda_j|)|E|^{\frac{1}{n}} \le \mathcal{A}_n \sup_{A_j\in E} |\det(\lambda_1A_1+\cdots+\lambda_nA_n)|. \tag{1.3.21}$$

Corollary B. *There exists a finite constant $\mathcal{B}_n$ such that for any measurable compact convex set $E \subset \mathfrak{M}^{n\times n}$ of finite measure,*

$$|E|^{\frac{1}{n}} \le \mathcal{B}_n \sup_{A\in E} |\det(A)|. \tag{1.3.22}$$

See Section 3.3 for the proof of Corollary B.

Remarks 2.

1. One can easily check that

$$\sup_{A\in \mathrm{co}\{0,E\}} |\det(A)| = \sup_{A\in E} |\det(A)|.$$

This is because $|\det(\lambda A)| = \lambda^n|\det(A)|$ for any $\lambda \in [0,1]$, so we can always assume that $0 \in E$. Given a measurable $E \subset \mathfrak{M}^{n\times n}$, by scaling let $\widetilde{E} = rE$, $0 \neq r \in \mathbb{R}$, then

$$(|\widetilde{E}|)^{\frac{1}{n}} = (r^{n^2}|E|)^{\frac{1}{n}} = r^n|E|^{\frac{1}{n}},$$

and

$$\sup_{A\in\widetilde{E}} |\det(A)| = r^n \sup_{A\in E} |\det(A)|.$$

However, (1.3.22) is not translation invariant.

2. We use a counterexample to show that (1.3.22) fails without the convex condition. Take $n = 2$ as an example, and let

$$E = \{(a, b, c, d) : 0 \leq ad \leq 1, 0 \leq bc \leq 1, \text{ and } 1/N \leq a \leq N, 1/N \leq b \leq N\}.$$

Then we have

$$\sup_{A \in E} |\det(A)| = \sup_{A \in E} |\begin{vmatrix} a & c \\ b & d \end{vmatrix}| \leq 2.$$

and $|E| = (2 \ln N)^2$. Let $N \to \infty$, then it is contradicted to (1.3.22).

Remarks 3.

1. An open problem is what the best constants $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$ are. We prove in Chapter 3 that balls or ellipsoids are not their optimisers.

2. Note that inequalities of matrix type introduced in this part do not enjoy an obvious affine invariance. Nevertheless, there is an important action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathfrak{M}^{n\times n}(\mathbb{R})$ by premultiplication. That is, if $T \in \mathrm{GL}_n(\mathbb{R})$, $A \in \mathfrak{M}^{n\times n}(\mathbb{R})$ and $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$, then

$$\det(TA) = \det(T)\det(A)$$

and

$$|TE| = |\det(T)|^n |E|.$$

So both matrix inequalities in this paper are invariant under premultiplication by a matrix of unimodular determinant. We do not use the invariance of the entire problem under the action of left-multiplication by members of $\mathrm{SL}_n(\mathbb{R})$ but instead the facts which underly this invariance, i.e. that this action preserves determinants of individual matrices and preserves volumes of sets. It enters as a "catalyst" in order to obtain a measure theoretic consequence and its presence vanishes without trace.

1.4 Main Results of This Thesis

1.4.1 Technical Tools

One of the main purposes of this thesis is to establish a series of geometric inequalities related to fractional integration raised in Section 1.2 and to prove their sharp versions with L^p bounds in the Euclidean spaces. Furthermore, we go on to study the optimisers of these inequalities for characteristic functions and analyse the property of some matrix inequalities which examine the isodiametric problem in more general settings discussed in Section 1.3.

In order to find the optimisers of geometric functionals in the Euclidean spaces, we present some new rearrangement inequalities which are useful tools to determine their sharp versions. Classical rearrangement inequalities are extremely useful analytic tools (see [32], [34], [11], [9]). For example, rearrangement inequalities deduce the optimisers for the Hardy-Littlewood-Sobolev inequality that the extremal functions are spherically symmetric functions. Spherically symmetric functions are also the optimisers for the Sobolev inequality, Young's inequality. Classical rearrangement inequalities solve many other geometric problems. It gives the isoperimetric inequality that the surface area of a body decreases if replacing the body by its spherical rearrangement. In other examples classical rearrangement inequalities gives the Brunn-Minkowski inequality that the volume of the sum of two sets decreases if replacing the two sets by their spherical rearrangements.

1.4.2 Main Results

For the question (1.2.2), we obtain the bilinear form in metric spaces with geometric settings.

Theorem 1.4.1. *Let $(M,\ d)$ be a metric space and μ a σ-finite, nonnegative Borel measure on M.*

(a) For any $x \in M$, $r > 0$, if μ satisfies for all ball $B(x,r) \subset M$

$$\mu(B(x,r)) \leq C_\alpha r^\alpha \tag{1.4.1}$$

with a finite constant C_α, $\alpha > 0$, then

$$\|f\|_{L^p(d\mu)}\ \|g\|_{L^q(d\mu)} \leq C_{p,q,\gamma} \sup_{s,t}\ f(s)g(t)d(s,t)^\gamma \tag{1.4.2}$$

holds for all nonnegative functions $f \in L^p(d\mu), g \in L^q(d\mu)$ for all $0 < p, q < \infty$, γ such that $\gamma = \alpha(\frac{1}{p} + \frac{1}{q})$.

(b) If inequality (1.4.2) holds for all nonnegative functions $f \in L^p(d\mu), g \in L^q(d\mu)$ for some $p, q > 0, \gamma > 0$, then (1.4.1) holds for all α such that

$$\alpha = \gamma(\frac{1}{p} + \frac{1}{q})^{-1}.$$

Combing with Gressman's work on multilinear determinant functionals [21], we prove inequality (1.2.3) is true in a real finite-dimensional Hilbert space with a certain geometric measure as follows (see Chapter 2 for definition and notation).

Theorem 1.4.2. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real finite-dimensional Hilbert space. Let μ be a σ-finite nonnegative Borel measure on H .*

(a) If μ satisfies for all ellipsoids B in H

$$\mu(B) \leq C_\alpha |B|_k^\alpha \tag{1.4.3}$$

with a finite constant C_α, $\alpha > 0$, $K \leq \dim H$, then

$$\prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)} \leq C_{p_j,k,\alpha} \sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \ldots, y_{k+1})^\gamma \tag{1.4.4}$$

holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$, for all $0 < p_j < \infty, \gamma$ which satisfy

$$\frac{1}{p_j} < \frac{\gamma}{k\alpha}, \quad \sum_{j=1}^{k+1} \frac{1}{p_j} = \frac{\gamma}{\alpha}, \quad 1 \leq j \leq k+1.$$

(b) If inequality (1.4.4) holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$, $j = 1, \ldots, k+1$, *for some* $p_j > 0$, $\gamma > 0$, *then (1.4.3) holds for all* α *such that*

$$\alpha = \gamma(\sum_{j=1}^{k+1} \frac{1}{p_j})^{-1}.$$

As for the second multilinear form (1.2.4), we prove the necessary and sufficient conditions for inequality (1.2.4) to hold in the trilinear case when $N = 3$. However, our method does not work for other multilinear cases. Applying Beckner's work on geometric inequalities in Fourier analysis [3], we obtain sufficient conditions for inequality (1.2.4) to hold for all N as follows.

Theorem 1.4.3. *Let* $r_{ij} > 0$ *and* $r_{ij} = r_{ji}$. *Let* f_j *be nonnegative measurable functions defined on* $\mathbb{R}^n$, *then*

$$\prod_{j=1}^{3} \|f_j\|_{p_j} \le C_{p_j, r_{ij}, n} \sup_{y_j} \prod_{j=1}^{3} f_j(y_j) \prod_{1 \le i < j \le 3} |y_i - y_j|^{r_{ij}} \tag{1.4.5}$$

holds, if and only if p_j *satisfy*

$$\sum_{j=1}^{3} \frac{1}{p_j} = \frac{1}{n}(r_{12} + r_{13} + r_{23}) \ , \ \tfrac{1}{p_j} < \tfrac{1}{n} \sum_{i \ne j} r_{ij} \ \ \text{for every } j.$$

Theorem 1.4.4. *Let* $r_{ij} > 0$ *and* $r_{ij} = r_{ji}$. *Let* f_j *be nonnegative measurable functions defined on* $\mathbb{R}^n$, $1 \le j \le N$. *Then*

$$\prod_{j=1}^{N} \|f_j\|_{p_j} \le C_{p_j, r_{ij}, n, N} \sup_{y_j} \prod_{j=1}^{N} f_j(y_j) \prod_{1 \le i < j \le N} |y_i - y_j|^{r_{ij}}, \tag{1.4.6}$$

holds for any $0 < p_j < \infty$ *satisfying*

$$\frac{1}{p_j} = \frac{1}{2n} \sum_{i \ne j} r_{ij}, \quad \sum_{j=1}^{N} \frac{1}{p_j} = \frac{1}{n} \sum_{1 \le i < j \le N} r_{ij}. \tag{1.4.7}$$

As discussed in Subsection 1.4.1, in order to determine the sharp constants and optimisers we need take advantage of some technial tools called rearrangement inequalities. Here we list two important rearrangement inequalities which will be studied in Chapter 3.

Theorem 1.4.5. *(bilinear rearrangement inequality) Let* f, g *be nonnegative measurable functions defined on* $\mathbb{R}^n$. *Then*

$$\sup_{s,t} f^*(s) g^*(t) |s - t| \le \sup_{x,y} f(x) g(y) |x - y|, \tag{1.4.8}$$

where f^* *is its symmetric decreasing rearrangement defined as*

$$f^*(x) := \int_0^\infty \chi_{\{f > t\}^*}(x) dt.$$

Theorem 1.4.6. *(multilinear rearrangement inequality) Let f_j be nonnegative measurable functions defined on $\mathbb{R}^n$, $1 \le j \le n+1$. Then*

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^*(y_j) \det(y_1, \ldots, y_{n+1}) \le \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1}). \tag{1.4.9}$$

In Chapter 4, by competing symmetries together with rearrangement inequalities (1.4.8)-(1.4.9) we obtain the sharp form of the geometric inequalities (1.4.2) and (1.4.4) with L^p bounds in the Euclidean spaces as follows. The optimisers of (1.4.2) and (1.4.4) for characteristic functions that is the problem raised at the beginning of Section 1.3 will be focused on in Chapter 3 as applications of rearrangement inequalities.

Theorem 1.4.7. *Let $0 < p < \infty$ and f, g be in $L^p(\mathbb{R}^n)$. For the geometric inequality*

$$\|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \le C_{p,n} \sup_{x,y} f(x)g(y)|x-y|^{\frac{2n}{p}}, \tag{1.4.10}$$

the minimum constant $C_{p,n} = 2^{-\frac{2n}{p}} |\mathbb{S}^n|^{\frac{2}{p}}$ which is obtained for $f = const \cdot h$, and $g = const \cdot h$, where $|\mathbb{S}^n|$ is the surface area of the unit sphere $\mathbb{S}^n$, and

$$h(x) = (1+|x|^2)^{-\frac{n}{p}}.$$

Theorem 1.4.8. *Let $0 < p < \infty$ and f_j be in $L^p(\mathbb{R}^n)$. For multilinear geometric inequality*

$$\prod_{j=1}^{n+1} \|f_j\|_{L^p(\mathbb{R}^n)} \le C_{p,n} \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{\gamma} \tag{1.4.11}$$

with $\gamma = \frac{n+1}{p}$, the minimum constant is $C_{p,n} = (\frac{1}{2}|\mathbb{S}^n|)^{\frac{n+1}{p}}$ which is obtained when $f_j = const \cdot h$, $1 \le j \le n+1$, where $|\mathbb{S}^n|$ is the surface area of the unit sphere $\mathbb{S}^n$, and

$$h(x) = (1+|x|^2)^{-\frac{n+1}{2p}}.$$

1.5 Outline of This Thesis

This thesis is divided into five chapters. In Chapter 2, we will deal with various types of geometric inequalities stated in Section 1.2. Chapter 3 is devoted to the study of some new rearrangement inequalities as introduced in technical tools above. In the last section we prove the matrix type of geometric inequalities discussed in Section 1.3, mainly applying the rearrangment inequality. Specifically, we prove the Main Theorem and Corollary A, B which are of the principal results of this thesis. In Chapter 4, the sharp versions of bilinear form and multilinear form in the Euclidean spaces settings with L^p bounds are obtained, and their optimisers for characteristic functions will be contained in Chapter 3 as the application of rearrangement inequalities. The results in Chapter 2 and Chapter 4 have been obtained in [12], and somes of the results in Chapter 3 appear in [13]. Chapter 5 is about the regularity of Hardy-Littlewood maximal functions at the endpoint case in one dimension. This work is collaborated with Professors F. Liu and H. Wu at Xiamen University (see [35]). In the first section we start by introducing the history of regularity problems of Hardy-Littlewood maximal functions. In the rest sections, by analyzing the local maximum point of Hardy-Littlewood maximal functions we give a different proof to improve Tanaka's result in [46].

Chapter 2

Geometric Inequalities Related to Fractional Integration

In this chapter, we will investigate bilinear geometric inequalities in Section 2.1 and multilinear geometric inequalities in Section 2.2. As mentioned in the introduction, we will give two possible multilinear versions. One is the determinant form which is presented in Subsection 2.2.2, and the other is the product form in Subsection 2.2.3.

2.1 Bilinear Geometric Inequalities

In this section, we find the bilinear geometric inequalities depends strongly on the measure of balls shown in Theorem 2.1.1.

Let $(M,\ d)$ be a metric space and μ a σ-finite nonnegative Borel measure on M. Let f, g be nonnegative measurable functions defined on M. We consider the two conditions:

(i) For any $x \in M, r > 0$, μ satisfies

$$\mu(B(x,r)) \leq C_\alpha r^\alpha \tag{2.1.1}$$

with a finite constant C_α, $\alpha > 0$.

(ii)

$$\|f\|_{L^p(d\mu)}\ \|g\|_{L^q(d\mu)} \leq C_{p,q,\gamma} \sup_{s,t}\ f(s)g(t)d(s,t)^\gamma \tag{2.1.2}$$

holds for all nonnegative functions $f \in L^p(d\mu), g \in L^q(d\mu)$ with a finite constant $C_{p,q,\gamma}$ independent of the funtions f, g.

The main results are as follows.

Theorem 2.1.1. *Let $(M,\ d)$ be a metric space and μ a σ-finite, nonnegative Borel measure on M.*

(a) If condition (i) holds, then (ii) holds for all nonnegative functions $f \in L^p(d\mu), g \in L^q(d\mu)$ for all $0 < p, q < \infty$, γ such that $\gamma = \alpha(\frac{1}{p} + \frac{1}{q})$.

(b) If condition (ii) holds for all nonnegative functions $f \in L^p(d\mu), g \in L^q(d\mu)$ for some $p, q > 0, \gamma > 0$, then condition (i) holds for all α such that

$$\alpha = \gamma(\frac{1}{p} + \frac{1}{q})^{-1}.$$

We begin by studying an endpoint case of (2.1.2) in the following Lemma 2.1.2, before studying Theorem 2.1.1 itself.

Lemma 2.1.2. *Let f, g be nonnegative measurable functions defined on a metric space $(M,\ d)$ with the σ-finite and nonnegative Borel measure μ which satisfies $\mu(B(x,r)) \le C_\alpha r^\alpha$ for any $x \in M, r > 0$. Then for all $0 < p, q < \infty$ we have*

$$\|f\|_{L^{p,\infty}(d\mu)}\|g\|_{L^\infty(d\mu)} \le C_{p,\alpha} \sup_{s,t} f(s)g(t)d(s,t)^{\frac{\alpha}{p}}. \tag{2.1.3}$$

$$\|f\|_{L^\infty(d\mu)}\ \|g\|_{L^{q,\infty}(d\mu)} \le C_{q,\alpha} \sup_{s,t} f(s)g(t)d(s,t)^{\frac{\alpha}{q}}. \tag{2.1.4}$$

Proof. If $\sup\limits_{s,t} f(s)g(t)d(s,t)^{\frac{\alpha}{p}} = \infty$, then the inequality (2.1.3) is trivial.
If $\sup\limits_{s,t} f(s)g(t)d(s,t)^{\frac{\alpha}{p}} = A < \infty$, there exists a measure zero set $E \subset M \times M$, $\mu \otimes \mu(E) = 0$, such that for any $(s,t) \in (M \times M) \setminus E$

$$f(s)g(t)d(s,t)^{\frac{\alpha}{p}} \le A. \tag{2.1.5}$$

Note that for any $\varepsilon > 0$, there exists $F \subset M$, $\mu(F) > 0$, such that for all $t \in F$

$$g(t) > \|g\|_{L^\infty(d\mu)} - \varepsilon. \tag{2.1.6}$$

It follows from (2.1.5) and (2.1.6) that for all $(s,t) \in (M \times F) \setminus E$

$$f(s) \le \frac{A}{d(s,t)^{\frac{\alpha}{p}}(\|g\|_\infty - \varepsilon)}.$$

So we can choose a $t \in F$ such that for any $\beta > 0$,

$$\mu(\{s : f(s) > \beta\}) \le \mu(\{s : d(s,t)^{\frac{\alpha}{p}} < \frac{A}{\beta(\|g\|_\infty - \varepsilon)}\}).$$

This is because $\mu \otimes \mu(E) = 0$ implies that for almost every $t \in M$,

$$\mu(\{s \in M : (s,t) \in E\}) = 0.$$

And since $\mu(F) > 0$, we can find $t \in F$ such that $(s,t) \in (M \times F) \setminus E$ for almost every $s \in M$.

Calculate the weak L^p "norm" of f,

$$\begin{aligned}
\|f\|_{L^{p,\infty}(d\mu)} &= \sup_{\beta>0} \beta\ \mu(\{s : f(s) > \beta\})^{\frac{1}{p}} \\
&\leq \sup_{\beta>0} \beta\ \mu(\{s : d(s,t)^{\frac{k\alpha}{p}} < \frac{A}{\beta(\|g\|_\infty - \varepsilon)}\})^{\frac{1}{p}} \\
&= \sup_{\beta>0} \beta\ \mu(\{s : d(s,t) < (\frac{A}{\beta(\|g\|_\infty - \varepsilon)})^{\frac{p}{\alpha}}\})^{\frac{1}{p}}.
\end{aligned}$$

Since $\mu(B(x,r)) \leq C_\alpha r^\alpha$ for any $x \in M, r > 0$,

$$\mu(\{s : d(s,t) < r\}) \leq C_\alpha r^\alpha.$$

Hence

$$\mu(\{s : d(s,t) < (\frac{A}{\beta(\|g\|_\infty - \varepsilon)})^{\frac{p}{\alpha}}\}) \leq C_\alpha(\frac{A}{\beta(\|g\|_\infty - \varepsilon)})^p.$$

Then we get

$$\begin{aligned}
\|f\|_{L^{p,\infty}(d\mu)} &\leq \sup_{\beta>0} \beta\ \mu(\{s : d(s,t) < (\frac{A}{\beta(\|g\|_\infty - \varepsilon)})^{\frac{p}{\alpha}}\})^{\frac{1}{p}} \\
&\leq C_\alpha^{\frac{1}{p}} \sup_{\beta>0} \beta\ \frac{A}{\beta(\|g\|_\infty - \varepsilon)} \\
&= C_\alpha^{\frac{1}{p}} \frac{A}{\|g\|_\infty - \varepsilon},
\end{aligned}$$

that is

$$\|f\|_{L^{p,\infty}(d\mu)}(\|g\|_{L^\infty(d\mu)} - \varepsilon) \leq C_\alpha^{\frac{1}{p}} A.$$

Let $\varepsilon \to 0$, we have

$$\|f\|_{L^{p,\infty}(d\mu)}\|g\|_{L^\infty(d\mu)} \leq C_\alpha^{\frac{1}{p}} A = C_\alpha^{\frac{1}{p}} \sup_{s,t} f(s)g(t)d(s,t)^{\frac{\alpha}{p}}.$$

Likewise,

$$\|f\|_{L^\infty(d\mu)}\ \|g\|_{L^{q,\infty}(d\mu)} \leq C_\alpha^{\frac{1}{q}} \sup_{s,t} f(s)g(t)d(s,t)^{\frac{\alpha}{q}}.$$

□

Now we are in a position to prove Theorem 2.1.1.

Proof of Theorem 2.1.1

(a) Suppose condition (i) holds, that is $\mu(B(t,r)) \leq Cr^\alpha$ holds for any $t \in M, r > 0$. Let $m = \frac{1}{\frac{1}{p}+\frac{1}{q}}$, so $m < p,\ q < \infty$.

Then by the layer cake representation

$$\begin{aligned}
\|f\|^p_{L^p(d\mu)} &= p\int_0^\infty \beta^{p-1}\mu(\{s: f(s)>\beta\})\ d\beta \\
&= p\int_0^{\|f\|_{L^\infty(d\mu)}} \beta^{p-1}\mu(\{s: f(s)>\beta\})\ d\beta + p\int_{\|f\|_{L^\infty(d\mu)}}^\infty \beta^{p-1}\mu(\{s: f(s)>\beta\})\ d\beta \\
&= p\int_0^{\|f\|_{L^\infty(d\mu)}} \beta^{p-m-1}\beta^m\mu(\{s: |f(s)|>\beta\})\ d\beta \\
&\le p\ \|f\|^m_{L^{m,\infty}(d\mu)}\int_0^{\|f\|_{L^\infty(d\mu)}} \beta^{p-m-1}\ d\beta \\
&= \frac{p}{p-m}\ \|f\|^m_{L^{m,\infty}(d\mu)}\|f\|^{p-m}_{L^\infty(d\mu)},
\end{aligned}$$

which means for f in $L^{m,\infty}(d\mu)\cap L^\infty(d\mu)$, we have $f\in L^p(d\mu)$, and

$$\|f\|_{L^p(d\mu)} \le (\frac{p}{p-m})^{\frac{1}{p}}\ \|f\|^{\frac{m}{p}}_{L^{m,\infty}(d\mu)}\|f\|^{1-\frac{m}{p}}_{L^\infty(d\mu)}. \tag{2.1.7}$$

Meanwhile if g is in $L^{m,\infty}(d\mu)\cap L^\infty(d\mu)$, then $g\in L^q(d\mu)$, and

$$\|g\|_{L^q(d\mu)} \le (\frac{q}{q-m})^{\frac{1}{q}}\ \|g\|^{\frac{m}{q}}_{L^{m,\infty}(d\mu)}\|g\|^{1-\frac{m}{q}}_{L^\infty(d\mu)}. \tag{2.1.8}$$

Since simple functions are in $L^{m,\infty}(d\mu)\cap L^\infty(d\mu)$, we can apply Lemma 2.1.2 for simple functions f,g. Inequalities (2.1.7) and (2.1.8) indicate

$$\begin{aligned}
\|f\|_{L^p(d\mu)}\ \|g\|_{L^q(d\mu)} &\le (\frac{p}{p-m})^{\frac{1}{p}}\ (\frac{q}{q-m})^{\frac{1}{q}}\|f\|^{\frac{m}{p}}_{m,\infty}\|f\|^{1-\frac{m}{p}}_\infty\|g\|^{\frac{m}{q}}_{m,\infty}\|g\|^{1-\frac{m}{q}}_\infty \\
&= (\frac{p}{p-m})^{\frac{1}{p}}\ (\frac{q}{q-m})^{\frac{1}{q}}(\|f\|^{\frac{m}{p}}_{m,\infty}\|g\|^{1-\frac{m}{q}}_\infty)(\|g\|^{\frac{m}{q}}_{m,\infty}\|f\|^{1-\frac{m}{p}}_\infty).
\end{aligned}$$

It follows from Lemma 2.1.2 that

$$\|f\|^{\frac{m}{p}}_{m,\infty}\|g\|^{1-\frac{m}{q}}_\infty = \|f\|^{\frac{m}{p}}_{m,\infty}\|g\|^{\frac{m}{p}}_\infty \le C_\alpha^{\frac{1}{p}}\sup_{s,t}\ (f(s)g(t)d(s,t)^{\frac{\alpha}{m}})^{\frac{m}{p}},$$

and

$$\|g\|^{\frac{m}{q}}_{m,\infty}\|f\|^{1-\frac{m}{p}}_\infty = \|g\|^{\frac{m}{q}}_{m,\infty}\|f\|^{\frac{m}{q}}_\infty \le C_\alpha^{\frac{1}{q}}\sup_{s,t}\ (f(s)g(t)d(s,t)^{\frac{\alpha}{m}})^{\frac{m}{q}}.$$

Therefore

$$\begin{aligned}
\|f\|_{L^p(d\mu)}\ \|g\|_{L^q(d\mu)} &\le C_\alpha^{\frac{1}{p}+\frac{1}{q}}(\frac{p}{p-m})^{\frac{1}{p}}\ (\frac{q}{q-m})^{\frac{1}{q}}\sup_{s,t}(f(s)g(t)d(s,t)^{\frac{\alpha}{m}})^{\frac{m}{p}}\sup_{s,t}(f(s)g(t)d(s,t)^{\frac{\alpha}{m}})^{\frac{m}{q}} \\
&= C_\alpha^{\frac{1}{p}+\frac{1}{q}}(\frac{p+q}{p})^{\frac{1}{p}}(\frac{p+q}{q})^{\frac{1}{q}}\sup_{s,t}\ f(s)g(t)d(s,t)^{\frac{\alpha}{p}+\frac{\alpha}{q}}.
\end{aligned}$$

For general functions $f\in L^p(d\mu), g\in L^q(d\mu)$, there exist sequences of simple functions $\{f_n\}\uparrow f$, and $\{g_n\}\uparrow g$ as $n\to\infty$. Under the discussion above, we have already obtained that (2.1.2) holds for simple functions,

$$\|f_n\|_{L^p(d\mu)}\ \|g_n\|_{L^q(d\mu)} \leq C_{p,q,\alpha}\sup_{s,t} f_n(s)g_n(t)d(s,t)^{\alpha(\frac{1}{p}+\frac{1}{q})} \leq C_{p,q,\alpha}\sup_{s,t} f(s)g(t)d(s,t)^{\alpha(\frac{1}{p}+\frac{1}{q})}.$$

Then let $n \to \infty$, we have

$$\|f\|_{L^p(d\mu)}\ \|g\|_{L^q(d\mu)} \leq C_{p,q,\alpha}\sup_{s,t} f(s)g(t)d(s,t)^{\alpha(\frac{1}{p}+\frac{1}{q})}.$$

(b) Suppose $\|f\|_{L^p(d\mu)}\ \|g\|_{L^q(d\mu)} \leq C_{p,q,\gamma}\sup_{s,t} f(s)g(t)d(s,t)^{\gamma}$ holds for some $p,q > 0$, γ. For any $x \in M, r > 0$, let $f = g = \chi_{B(x,r)}$, then we have

$$\mu(B(x,r))^{\frac{1}{p}+\frac{1}{q}} \leq C_{p,q,\gamma}\sup_{s,t\in B(x,r)} d(s,t)^{\gamma}.$$

Together with the fact

$$\sup_{s,t\in B(x,r)} d(s,t) \leq 2r$$

we deduce that μ has the property

$$\mu(B(x,r)) \leq C_{\alpha} r^{\alpha},$$

where $\alpha = \gamma(\frac{1}{p}+\frac{1}{q})^{-1}$.

□

As an immediate consequence of Theorem 2.1.1, we have the bilinear analogue of inequality (2.1.2) in Euclidean spaces.

Corollary 2.1.3. *Let f, g be nonnegative measurable functions defined on $\mathbb{R}^n$ with Lebesgue measure, then for all $0 < p, q < \infty$, $\gamma > 0$ such that $\gamma = n(\frac{1}{p}+\frac{1}{q})$,*

$$\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} \leq C_{p,q,n}\sup_{s,t} f(s)g(t)|s-t|^{\gamma}. \tag{2.1.9}$$

On the other hand, it is not true that

$$\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{s,t} f(s)g(t)|s-t|^{\frac{n}{p}}. \tag{2.1.10}$$

holds for all $f \in L^p(\mathbb{R}^n), g \in L^\infty(\mathbb{R}^n)$.

Proof. (1) Observe that $|B(x,r)| \leq C_n r^n$ for any $x \in \mathbb{R}^n, r > 0$, then we can apply Theorem 2.1.1 to give (2.1.9). More precisely, γ here must be $n(\frac{1}{p}+\frac{1}{q})$ which follows from the homogeneity mentioned in the introduction.

(2) We use a counterexample to show that (2.1.10) fails.
For any positive N, let

$$f_N(s) = (1+|s|)^{-\frac{n}{p}}\chi_{(1\leq|s|\leq N)}, \quad g(t) = \chi_{(|t|\leq 1)}(t).$$

Then $\|g\|_{L^\infty(\mathbb{R}^n)} = 1$ and

$$\sup_{s,t} \; f_N(s)g(t) \; |s-t|^{\frac{n}{p}} = \sup_{s,t} \frac{|s-t|^{\frac{n}{p}}}{(1+|s|)^{\frac{n}{p}}} \; \chi_{(1\le|s|\le N)}(s) \; \chi_{(|t|\le 1)}(t)$$
$$\le \frac{(|s|+1)^{\frac{n}{p}}}{(1+|s|)^{\frac{n}{p}}} = 1.$$

While by polar coordinates

$$\|f_N\|^p_{L^p(\mathbb{R}^n)} = \int_{1\le|s|\le N} \frac{ds}{(1+|s|)^n} = C\int_1^N \frac{r^{n-1}}{(1+r)^n}dr.$$

Let $u = 1+r$ to make the change of variables

$$\int_1^N \frac{r^{n-1}}{(1+r)^n}dr = \int_2^{N+1} \frac{(u-1)^{n-1}}{u^n}du \ge \int_2^{N+1} \frac{1}{2^{n-1}}\frac{1}{u}du$$

and

$$\int_2^{N+1} \frac{1}{u}du = \ln(N+1) - \ln 2 \to \infty,$$

as $N \to \infty$.

□

The Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{C}^n \times \mathbb{R}$ with elements $x = (z,t)$, $y = (w,s)$ and the group operation

$$xy = (z_1,\ldots,z_n,t)(w_1,\ldots,w_n,s) = (z_1+w_1,\ldots,z_1+w_1,t+s+2\mathrm{Im}\sum_{j=1}^n z_j\overline{w_j}).$$

Haar measure on $\mathbb{H}^n$ is the usual Lebesgue measure $dx = dzdt$. The norm

$$|x| = |(z,t)| = (|z|^4+t^2)^{\frac{1}{4}} = [(\sum_{j=1}^n |z_j|^2)^2 + t^2]^{\frac{1}{4}}.$$

We define the distance $|x-y|$ as $|x-y| := |x^{-1}y|$, where

$$x^{-1} = (z,t)^{-1} = (-z_1,\ldots,-z_n,-t).$$

Clearly, it is symmetric $|x-y| = |y-x|$. Although it is a pseudometric, we still have the bilinear geometric inequality on $\mathbb{H}^n$ as follows.

Corollary 2.1.4. *Let f, g be nonnegative measurable functions defined on $\mathbb{H}^n$ with Haar measure, then for all $0 < p, q < \infty$, $\gamma > 0$ such that*

$$\|f\|_{L^p(\mathbb{H}^n)}\|g\|_{L^q(\mathbb{H}^n)} \le C_{p,q,n}\sup_{x,y} \; f(x)g(y)|x-y|^\gamma \tag{2.1.11}$$

holds, where $\gamma = (2n+2)(\frac{1}{p}+\frac{1}{q})$.

By the proof of Lemma 2.1.2, the geometric inequality (2.1.2) is only related to the measure of balls. It is known that on the Heisenberg group $\mathbb{H}^n$,

$$B(x,r) = xB(0,r), \quad |B(x,r)| = |B(0,r)| \le C_n r^{2n+2} \tag{2.1.12}$$

holds for all balls $B(x,r) := \{y \in \mathbb{H}^n : |x-y| < r\}$ in $\mathbb{H}^n$.

2.2 Multilinear Geometric Inequalities

2.2.1 Definition, Notation and Basic Properties

We first recall some definition, notation and lemmas which are all given in [21]. $(H, \langle\cdot,\cdot\rangle_H)$ is a real finite-dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$. For any positive integer $k \le \dim H$, we use $\det(y_1,\ldots,y_{k+1})$ to denote the square root of the determinant of the $k\times k$ Gram matrix $(a_{i,j})_{k\times k}$, where

$$a_{i,j} = \langle y_i - y_{k+1}, y_j - y_{k+1}\rangle_H.$$

Clearly, the Gram matrix $(a_{i,j})_{k\times k}$ is positive semidefinite, since $(a_{i,j})_{k\times k}$ can be written as $A'A$, where A is the matrix whose j-th column is $y_j - y_{k+1}$, and A' is the transpose of A. Thus throughout this section, $\det(y_1,\ldots,y_{k+1}) \ge 0$. Especially in Euclidean $\mathbb{R}^k$ space, the determinant of the matrix $(a_{i,j})_{k\times k}$ is the square of the volume of the parallelotope formed by the vectors $y_1,\ldots,y_{k+1}$. Thus, $\det(y_1,\ldots,y_{k+1})$ is also $k!$ times the Euclidean k-dimensional volume of the simplex with vertices $y_1,\ldots,y_{k+1}$.

Definition 2.2.1. *A subset $B \subset H$ is called an ellipsoid when it may be written as*

$$B \equiv \{x \in H : \sum_i \frac{|\langle x - x_0, \omega_i\rangle|^2}{l_i^2} \le 1\}$$

for some $x_0 \in H$, some orthonormal basis $\{\omega_i\}$ of H, and lengths $l_i \in [0,\infty]$.

For example, $\{(t,0,\ldots,0) : t \in \mathbb{R}\} \subset \mathbb{R}^n$ is an ellipsoid in $\mathbb{R}^n$. It could be written as

$$\frac{|\langle x, e_1\rangle|^2}{\infty} + \frac{|\langle x, e_2\rangle|^2}{0} + \cdots + \frac{|\langle x, e_n\rangle|^2}{0} \le 1,$$

where $l_1 = \infty, l_2 = 0, \ldots, l_n = 0$, $x_0 = 0$, and $\{e_1,\ldots,e_n\}$ are the standard orthonormal basis vectors for $\mathbb{R}^n$. The ellipsoid will be called centred when $x_0 = 0$. Given an ellipsoid $B \subset H$ and an integer k with $k \le \dim H$, denote

$$|B|_k = \sup\{l_{i_1}\ldots l_{i_k} : i_1 < i_2 < \cdots < i_k\},$$

which is called the k-content of B.

Definition 2.2.2. *Let μ be a σ-finite and nonnegative Borel measure on H. The measure μ is called k-curved with exponent $\alpha > 0$, if there exists a finite constant*

C_α such that

$$\mu(B) \le C_\alpha |B|_k^\alpha \tag{2.2.1}$$

holds for all ellipsoids B in H.

This kind of geometric measure describes the amount of mass of μ supported on k-dimensional subspaces of H. For instance, the Lebesgue measure in $\mathbb{R}^n$ is n-curved with exponent 1. It is k-curved with exponent $\frac{n}{k}$ as well for $k < n$. If we see the Lebesgue measure restricted on x_1 axis, it is 1-curved with exponent 1. It cannot be k-curved for $k \ge 2$. Let S be a hypersurface in $\mathbb{R}^n$ with non-vanishing Gaussian curvature, then its surface area measure μ_S is n-curved with exponent $\frac{n-1}{n+1}$.

In the following we recall some results of Gressman which all appear in [21].

Lemma 2.2.3. *Let μ be a σ-finite and nonnegative Borel measure such that (2.2.1) holds for all ellipsoids B in H. Then for any measurable sets $E_1, \ldots, E_k$ in H we have*

$$\mu\otimes\cdots\otimes\mu(\{(y_1, \ldots, y_k) \in E_1\times\cdots\times E_k : \det(0, y_1, \ldots, y_k) < \delta\}) \le C_{k,\alpha}\delta^\alpha \prod_{j=1}^{k} \mu(E_j)^{1-\frac{1}{k}}.$$

Lemma 2.2.4. *Under the above assumptions, for any centred ellipsoid B in H, we have*

$$\sup_{\substack{x_j \in B \\ j=1,\ldots,k}} \det(0, x_1, \ldots, x_k) \le C_k |B|_k, \tag{2.2.2}$$

where $|B|_k$ is the k-content of B.

Lemma 2.2.5. *Let f_j be nonnegative measurable functions defined on a real finite-dimensional Hilbert space H, and let μ be a σ-finite nonnegative Borel measure on H which satisfies inequality (2.2.1). Then for all $1 \le p_j \le \infty$ satisfying $\frac{1}{p_j} > 1 - \frac{\gamma}{k\alpha}$, $j = 1, \ldots, k+1$, $0 < \gamma < \alpha$, and $k + 1 - \sum_{j=1}^{k+1} \frac{1}{p_j} = \frac{\gamma}{\alpha}$,*

$$\int_{(H)^{k+1}} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \ldots, y_{k+1})^{-\gamma} d\mu(y_1) \cdots d\mu(y_{k+1}) \le C \prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)} \tag{2.2.3}$$

holds with a finite constant C independent of the functions f_j.

Later we will see the relationship between (2.2.3) and (2.2.5).

2.2.2 Determinant Forms of Multilinear Inequalities

The first kind of multilinear analogue of the geometric inequality we start to study is the determinant form. We find there is a strong link between the measure of ellipsoids and multilinear determinant inequalities as given in the following theorem, mainly discussing the two conditions with $1 \le k \le \dim H$ fixed:

(i) There exists a finite constant C_α such that for all ellipsoids B in H,

$$\mu(B) \leq C_\alpha |B|_k^\alpha. \tag{2.2.4}$$

(ii)

$$\prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)} \leq C \sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \ldots, y_{k+1})^\gamma \tag{2.2.5}$$

for all nonnegative functions $f_j \in L^{p_j}(d\mu), j = 1, \ldots, k+1$, where C is a finite constant independent of functions f_j which only depends on p_j, k, γ .

Theorem 2.2.6. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real finite-dimensional Hilbert space. Let μ be a σ-finite nonnegative Borel measure on H .*

(a) If condition (i) holds, then (ii) holds for all nonnegative functions$f_j \in L^{p_j}(d\mu)$, for all $0 < p_j < \infty, \gamma$ which satisfy

$$\tfrac{1}{p_j} < \tfrac{\gamma}{k\alpha},\ 1 \leq j \leq k+1, \text{ and } \sum_{j=1}^{k+1} \frac{1}{p_j} = \frac{\gamma}{\alpha}.$$

(b) If condition (ii) holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$, $j = 1, \ldots, k+1$, for some $p_j > 0$, $\gamma > 0$, then condition (i) holds for all α such that

$$\alpha = \gamma\Big(\sum_{j=1}^{k+1} \frac{1}{p_j}\Big)^{-1}.$$

If we consider the special case when $k = 1$, the condition (2.2.4) is equivalent to the condition (2.1.1): It is clear that (2.2.4) implies (2.1.1). Conversely, suppose $\mu(B(x,r)) \leq C_\alpha r^\alpha$ holds for any $x \in H, r > 0$. Given an ellipsoid K centred at x_0, clearly $K \subset B(x_0, |K|_1)$. Thus

$$\mu(K) \leq \mu(B(x_0, |E|_1)) \leq C_\alpha |E|_1^\alpha,$$

which gives that μ is 1-curved with exponent α.

When $k = 1$, inequality (2.2.5) becomes the bilinear form (2.1.2). In Section 2.1 we stated that

$$\|f_1\|_{L^{p_1}(d\mu)}\ \|f_2\|_{L^{p_2}(d\mu)} \leq C_{p_1,p_2,\gamma} \sup_{s,t}\ f_1(s) f_2(t) |s-t|^\gamma$$

holds for any $0 < p_1, p_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{\gamma}{\alpha}$. Note that the condition $\frac{1}{p_j} < \frac{\gamma}{k\alpha}, j = 1, 2$, in Theorem 2.2.6 (a) is automatic in this case, since $0 < p_1,\ p_2 < \infty$.

We begin by studying why condition (ii) implies condition (i).

Proof of Theorem 2.2.6 (b)

Let $f_j = \chi_B$, where $B \subset H$ is an ellipsoid centred at $x_0 \in H$, $j = 1, \ldots, k+1$.

Since condition (ii) holds for some $p_j, j = 1, \dots, k+1, \gamma$, then we have

$$\prod_{j=1}^{k+1} \|\chi_B\|_{L^{p_j}(d\mu)} \lesssim \sup_{y_j \in B} \det(y_1, \dots, y_{k+1})^\gamma,$$

that is,

$$\prod_{j=1}^{k+1} \mu(B)^{\frac{1}{p_j}} = \mu(B)^{\sum_{j=1}^{k+1} \frac{1}{p_j}} \lesssim \sup_{y_j \in B} \det(y_1, \dots, y_{k+1})^\gamma. \tag{2.2.6}$$

We use a fact that for any centred ellipsoid E, $E - E \subset 2E$. Suppose

$$E = \{x \in H : \sum_i \frac{|\langle x, \omega_i \rangle|^2}{l_i^2} \leq 1\}$$

where $\{\omega_i\}$ is the orthonormal basis of H. Let $y, z \in E$, since for every ω_i

$$|\langle y - z, \omega_i \rangle|^2 = |\langle y, \omega_i \rangle - \langle z, \omega_i \rangle|^2 \leq 2(|\langle y, \omega_i \rangle|^2 + |\langle z, \omega_i \rangle|^2),$$

it is easy to verify that

$$y - z \in 2E = \{x \in H : \sum_i \frac{|\langle x, \omega_i \rangle|^2}{(2l_i)^2} \leq 1\}.$$

Thus we have

$$B - B = (B - x_0) - (B - x_0) \subset 2(B - x_0).$$

Therefore, it follows form Lemma 2.2.4 that

$$\begin{aligned}
\sup_{y_j \in B} \det(y_1, y_2, \dots, y_k, y_{k+1}) &= \sup_{y_j \in B} \det(0, y_1 - y_{k+1}, y_2 - y_{k+1}, \dots, y_k - y_{k+1}) \\
&\leq \sup_{x_j \in 2(B - x_0)} \det(0, x_1, x_2, \dots, x_k) \\
&\leq 2^k C_k |B - x_0|_k = 2^k C_k |B|_k.
\end{aligned}$$

So

$$\sup_{y_j \in B} \det(y_1, \dots, y_{k+1})^\gamma \lesssim |B|_k^\gamma.$$

Together with (2.2.6), we conclude that

$$\mu(B)^{\sum_{j=1}^{k+1} \frac{1}{p_j}} \lesssim \sup_{y_j \in B} \det(y_1, \dots, y_{k+1})^\gamma \lesssim |B|_k^\gamma.$$

So $\mu(B) \lesssim |B|_k^\alpha$ with $\alpha = \gamma(\sum_{j=1}^{k+1} \frac{1}{p_j})^{-1}$.

$\square$

On the other hand, in order to see what inequality (2.2.5) will be like if μ

is k-curved with exponent α, we first investigate an endpoint case of (2.2.5) as follows.

Lemma 2.2.7. *Let f_j be measurable functions defined on real finite-dimensional Hilbert space H with the σ-finite and nonnegative Borel measure μ which satisfies $\mu(B) \le C_\alpha |B|_k^\alpha$ for all ellipsoids $B \subset H$.*
Then for any positive γ we have

$$\prod_{j=1}^{k} \|f_j\|_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)} \|f_{k+1}\|_{L^\infty(d\mu)} \le C_{k,\alpha,\gamma} \sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \dots, y_{k+1})^\gamma. \quad (2.2.7)$$

Likewise for each $1 \le l \le k+1$

$$\prod_{i \ne l} \|f_i\|_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)} \|f_l\|_{L^\infty(d\mu)} \le C_{k,\alpha,\gamma} \sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \dots, y_{k+1})^\gamma \quad (2.2.8)$$

holds by symmetry.

Proof. If $\sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \dots, y_{k+1})^\gamma = \infty$, the inequality (2.2.7) is trivial. Suppose $\sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \dots, y_{k+1})^\gamma = A < \infty$, then there exists measure zero set $E \subset H \times \cdots \times H$, $\mu \otimes \cdots \otimes \mu(E) = 0$, such that

$$\prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \dots, y_{k+1})^\gamma \le A, \quad (2.2.9)$$

for all $(y_1, \dots, y_{k+1}) \in (H \times \cdots \times H) \setminus E$. Note that for any $\varepsilon > 0$, there exists $F \subset H$ such that $\mu(F) > 0$, and for all $y_{k+1} \in F$

$$f_{k+1}(y_{k+1}) > \|f_{k+1}\|_\infty - \varepsilon. \quad (2.2.10)$$

From (2.2.9) and (2.2.10) it follows that for $(y_1, \dots, y_{k+1}) \in (H \times \cdots \times H \times F) \setminus E$,

$$f_1(y_1) \le \frac{A}{\|f_{k+1}\|_\infty - \varepsilon} \prod_{j=2}^{k} f_j(y_j)^{-1} \det(y_1, \dots, y_{k+1})^{-\gamma}. \quad (2.2.11)$$

For any positive α_j, denote $C_j = \{y_j : f_j(y_j) > \alpha_j\}$, $j = 1, \dots, k$. Note that $\mu \otimes \cdots \otimes \mu(E) = 0$, which implies that for almost every $y_{k+1} \in H$,

$$\mu \otimes \cdots \otimes \mu(\{(y_1, \dots, y_k) : (y_1, \dots, y_k, y_{k+1}) \in E\}) = 0.$$

Denote $\{(y_1, \dots, y_k) : (y_1, \dots, y_k, y_{k+1}) \in E\}$ by $G_{y_{k+1}} \subset H^k$. Since $\mu(F) > 0$, we can choose a $y_{k+1} \in F$ such that $\mu \otimes \cdots \otimes \mu(G_{y_{k+1}}) = 0$, and for all $(y_1, \dots, y_k) \in H^k \setminus G_{y_{k+1}}$

$$(y_1, \dots, y_k, y_{k+1}) \in (H \times \cdots \times H \times F) \setminus E.$$

Since $\mu \otimes \cdots \otimes \mu(G_{y_{k+1}}) = 0$, for almost every $y_1 \in H$

$$\mu \otimes \cdots \otimes \mu(\{(y_2, \ldots, y_k) \in H^{k-1} : (y_1, y_2, \ldots, y_k) \in G_{y_{k+1}}\}) = 0.$$

That is to say, for almost every y_1, almost every $(y_2, \ldots, y_k) \in H^{k-1}$

$$(y_1, y_2, \ldots, y_k) \in (H \times \cdots \times H \times F) \setminus E.$$

Therefore, together with (2.2.11) implies that for any $\alpha_1 > 0$

$$\begin{aligned}
&\mu(\{y_1 : f_1(y_1) > \alpha_1\}) \\
\leq &\mu(\{y_1 : \det(y_1, \ldots, y_{k+1})^\gamma < \frac{A}{\alpha_1(\|f_{k+1}\|_\infty - \varepsilon)} \prod_{j=2}^k f_j(y_j)^{-1},\ (y_2, \ldots, y_k) \in H^{k-1}\ a.e.\ \}).
\end{aligned}$$

Denote $\frac{A}{\|f_{k+1}\|_\infty - \varepsilon}$ by B. Due to the definition of C_j, we get for any $\alpha_1 > 0$,

$$\begin{aligned}
&\mu(\{y_1 : f_1(y_1) > \alpha_1\}) \\
&\leq \mu(\{y_1 \in C_1 : \det(y_1, \ldots, y_{k+1})^\gamma < B\alpha_1^{-1} \prod_{j=2}^k \alpha_j^{-1},\ (y_2, \ldots, y_k) \in C_2 \times \cdots \times C_k\ a.e.\ \}) \\
&\leq \mu(\{y_1 \in C_1 : \det(y_1, \ldots, y_{k+1}) < B^{\frac{1}{\gamma}} \prod_{j=1}^k \alpha_j^{-\frac{1}{\gamma}},\ (y_2, \ldots, y_k) \in C_2 \times \cdots \times C_k\ a.e.\ \}).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mu(\{y_1 : f_1(y_1) > \alpha_1\}) \mu \otimes \cdots \otimes \mu(\{(y_2, \ldots, y_k) : (y_2, \ldots, y_k) \in C_2 \times \cdots \times C_k\ a.e.\}) \\
&= \mu \otimes \cdots \otimes \mu(\{(y_1, \ldots, y_k) \in C_1 \times \cdots \times C_k : \det(y_1, \ldots, y_{k+1}) < B^{\frac{1}{\gamma}} \prod_{j=1}^k \alpha_j^{-\frac{1}{\gamma}}\}).
\end{aligned}$$

Denote $B^{\frac{1}{\gamma}} \prod_{j=1}^k \alpha_j^{-\frac{1}{\gamma}}$ by M, then it follows from Lemma 2.2.1 that

$$\begin{aligned}
&\mu(C_1) \times \cdots \times \mu(C_k) \\
&\leq \mu \otimes \cdots \otimes \mu(\{(y_1, \ldots, y_k) \in C_1 \times \cdots \times C_k : \det(y_1, \ldots, y_{k+1}) < M\}) \\
&= \mu \otimes \cdots \otimes \mu(\{(y_1, \ldots, y_k) \in C_1 \times \cdots \times C_k : \det(0, y_1 - y_{k+1}, \ldots, y_k - y_{k+1}) < M\}) \\
&\leq C_{k,\alpha} M^\alpha \prod_{j=1}^k \mu(C_j)^{1-\frac{1}{k}}.
\end{aligned}$$

Therefore,

$$\prod_{j=1}^k \mu(C_j)^{\frac{1}{k}} \leq C_{k,\alpha} M^\alpha.$$

That is,

$$\prod_{j=1}^{k}\mu(C_j)^{\frac{1}{k}} \le C_{k,\alpha}B^{\frac{\alpha}{\gamma}}\prod_{j=1}^{k}\alpha_j^{-\frac{\alpha}{\gamma}} \le C_{k,\alpha}(\frac{A}{\|f_{k+1}\|_\infty-\varepsilon})^{\frac{\alpha}{\gamma}}\prod_{j=1}^{k}\alpha_j^{-\frac{\alpha}{\gamma}},$$

which implies for any $\alpha_j>0$,

$$\prod_{j=1}^{k}\alpha_j\mu(C_j)^{\frac{\gamma}{k\alpha}} \le C_{k,\alpha}^{\frac{\gamma}{\alpha}}\frac{A}{\|f_{k+1}\|_\infty-\varepsilon}.$$

Let $\varepsilon\to 0$, we get for any $\alpha_j>0$

$$\prod_{j=1}^{k}\alpha_j\mu(C_j)^{\frac{\gamma}{k\alpha}} \le C_{k,\alpha}^{\frac{\gamma}{\alpha}}\frac{A}{\|f_{k+1}\|_\infty}. \tag{2.2.12}$$

Since α_j are arbitrary, this allows us to take the infimum over all $\alpha_j>0$ on (2.2.12), $j=1,\dots,k+1$, which gives

$$\prod_{j=1}^{k}\|f_j\|_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)}\|f_{k+1}\|_{L^\infty(d\mu)} \le C_{k,\alpha}^{\frac{\gamma}{\alpha}}A = C_{k,\alpha}^{\frac{\gamma}{\alpha}}\sup_{y_j}\prod_{j=1}^{k+1}f_j(y_j)\det(y_1,\dots,y_{k+1})^\gamma.$$

This proves the endpoint case (2.2.7). Meanwhile by symmetry (2.2.8) holds.

□

Proof of Theorem 2.2.6 (a)

For any general $f_j\in L^{p_j}(d\mu)$, there exist sequences of simple functions $\{f_{jn}\}\uparrow f_j$ as $n\to\infty$. We apply Lemma 2.2.5 for simple functions f_{jn} , this is because simple functions are in $L^{\frac{k\alpha}{\gamma},\infty}(d\mu)\cap L^\infty(d\mu)$. For each $1\le j\le k+1$, for every n, we have

$$\prod_{i\neq j}\|f_{in}\|_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)}\|f_{jn}\|_{L^\infty(d\mu)} \lesssim \sup_{y_j}\prod_{j=1}^{k+1}f_{jn}(y_j)\det(y_1,\dots,y_{k+1})^\gamma. \tag{2.2.13}$$

Based on this, by the layer cake representation it is easy to obtain that for $\frac{1}{p_j}<\frac{\gamma}{k\alpha}$, $f_{jn}\in L^{p_j}(d\mu)$ and

$$\|f_{jn}\|_{L^{p_j}(d\mu)} \lesssim \|f_{jn}\|_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)}^{\frac{k\alpha}{\gamma p_j}}\|f_{jn}\|_{L^\infty(d\mu)}^{1-\frac{k\alpha}{\gamma p_j}}. \tag{2.2.14}$$

We assume that for every n

$$\sup_{y_j}\prod_{j=1}^{k+1}f_{jn}(y_j)\det(y_1,\dots,y_{k+1})^\gamma = A_n<\infty,$$

then from (2.2.13), (2.2.14) and $\sum_{j=1}^{k+1}\frac{1}{p_j}=\frac{\gamma}{\alpha}$ it follows that

$$\begin{aligned}
\prod_{j=1}^{k+1}\|f_{jn}\|_{L^{p_j}(d\mu)} &\lesssim \prod_{j=1}^{k+1}\|f_{jn}\|^{\frac{k\alpha}{\gamma p_j}}_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)}\|f_{jn}\|^{1-\frac{k\alpha}{\gamma p_j}}_{L^\infty(d\mu)}\\
&=\prod_{j=1}^{k+1}(\prod_{i\neq j}\|f_{in}\|_{L^{\frac{k\alpha}{\gamma},\infty}(d\mu)}\|f_{jn}\|_{L^\infty(d\mu)})^{1-\frac{k\alpha}{\gamma p_j}}\\
&\lesssim \prod_{j=1}^{k+1}(\sup_{y_j}\prod_{j=1}^{k+1}f_{jn}(y_j)\det(y_1,\dots,y_{k+1})^\gamma)^{1-\frac{k\alpha}{\gamma p_j}}\\
&=\prod_{j=1}^{k+1}A_n^{1-\frac{k\alpha}{\gamma p_j}}.
\end{aligned}$$

Note that $\sum_{j=1}^{k+1}(1-\frac{k\alpha}{\gamma p_j})=1$, since $\sum_{j=1}^{k+1}\frac{1}{p_j}=\frac{\gamma}{\alpha}$. Hence,

$$\prod_{j=1}^{k+1}\|f_{jn}\|_{L^{p_j}(d\mu)}\lesssim\prod_{j=1}^{k+1}A_n^{1-\frac{k\alpha}{\gamma p_j}}=A_n\equiv\sup_{y_j}\prod_{j=1}^{k+1}f_{jn}(y_j)\det(y_1,\dots,y_{k+1})^\gamma.$$

Therefore, for every n

$$\prod_{j=1}^{k+1}\|f_{jn}\|_{L^{p_j}(d\mu)}\leq\sup_{y_j}\prod_{j=1}^{k+1}f_{jn}(y_j)\det(y_1,\dots,y_{k+1})^\gamma\leq\sup_{y_j}\prod_{j=1}^{k+1}f_j(y_j)\det(y_1,\dots,y_{k+1})^\gamma.$$

Let $n\to\infty$ to deduce that

$$\prod_{j=1}^{k+1}\|f_j\|_{L^{p_j}(d\mu)}\lesssim\sup_{y_j}\prod_{j=1}^{k+1}f_j(y_j)\det(y_1,\dots,y_{k+1})^\gamma.$$

This completes the proof of this theorem.

□

We shall now present an alternative method to show that condition (i) implies condition (ii), mainly applying Gressman's result Lemma 2.2.5 above.

Alternative proof of Theorem 2.2.6 (a)

Let $q_j\in(0,\infty)$, $p_j\in(1,\infty)$, $j=1,\dots,k+1$, then

$$\sup_{y_j}\prod_{j=1}^{k+1}f_j(y_j)\det(y_1,\dots,y_{n+1})^\gamma=[\sup_{y_j}\prod_{j=1}^{k+1}f_j(y_j)^{\frac{q_j}{p_j}}\det(y_1,\dots,y_{n+1})^{\frac{\gamma q_j}{p_j}}]^{\frac{p_j}{q_j}},$$

and for each j,

$$\|f_j\|_{L^{q_j}(d\mu)}=\|f_j^{\frac{q_j}{p_j}}\|^{\frac{p_j}{q_j}}_{L^{p_j}(d\mu)}.$$

Thus if (2.2.5) holds for all $p_j \in (1, \infty)$ satisfying

$$\frac{1}{p_j} < \frac{\gamma}{k\alpha}, 1 \leq j \leq k+1, \ \sum_{j=1}^{k+1} \frac{1}{p_j} = \frac{\gamma}{\alpha}, \tag{2.2.15}$$

then (2.2.5) holds for all $0 < q_j < \infty$ satisfying

$$\tfrac{1}{q_j} < \tfrac{\gamma}{k\alpha}, \ 1 \leq j \leq k+1, \text{ and } \sum_{j=1}^{k+1} \frac{1}{q_j} = \frac{\gamma}{\alpha}.$$

Thus it suffices to show (2.2.5) for all $p_j \in (1, \infty)$ satisfying condition (2.2.15). Suppose

$$\sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^\gamma = A < \infty.$$

We can write

$$\begin{aligned}
&\|f_1\|_{p_1}^{p_1} \cdots \|f_{k+1}\|_{p_{k+1}}^{p_{k+1}} \\
&= \int_H \cdots \int_H \prod_{j=1}^{k+1} f_j(y_j)^{p_j} \ d\mu(y_1) \ldots d\mu(y_{k+1}) \\
&= \int_{(H)^{k+1}} \prod_{j=1}^{k+1} f_j(y_j)^{\frac{1}{k+2}} \det(y_1, \ldots, y_{k+1})^{\frac{\gamma}{k+2}} \prod_{j=1}^{k+1} f_j(y_j)^{p_j - \frac{1}{k+2}} \\
&\qquad \det(y_1, \ldots, y_{k+1})^{-\frac{\gamma}{k+2}} \ d\mu(y_1) \ldots d\mu(y_{k+1}).
\end{aligned}$$

Since

$$\sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \ldots, y_{k+1})^\gamma = A,$$

$$\|f_1\|_{p_1}^{p_1} \cdots \|f_{k+1}\|_{p_{k+1}}^{p_{k+1}} \leq A^{\frac{1}{k+2}} \int_{(H)^{k+1}} \prod_{j=1}^{k+1} f_j(y_j)^{p_j - \frac{1}{k+2}} \det(y_1, \ldots, y_{k+1})^{-\frac{\gamma}{k+2}} d\mu(y_1) \ldots d\mu(y_{k+1}).$$

From the condition

$$\frac{\gamma}{\alpha} = \sum_{j=1}^{k+1} \frac{1}{p_j} < k+2,$$

we have $0 < \frac{\gamma}{k+2} < \alpha$. Let $\frac{1}{\sigma_j} = 1 - \frac{1}{p_j(k+2)} \in (0, 1)$. It is easy to check that for each j

$$\frac{1}{\sigma_j} > 1 - \frac{\gamma}{k\alpha(k+2)}, \ k+1 - \sum_{j=1}^{k+1} \frac{1}{\sigma_j} = \frac{\gamma}{\alpha(k+2)}.$$

This allows us to apply Lemma 2.2.5 to get

$$\int_{(H)^{k+1}} \prod_{j=1}^{k+1} f_j(y_j)^{p_j - \frac{1}{k+2}} \det(y_1, \ldots, y_{k+1})^{-\frac{\gamma}{k+2}} d\mu(y_1) \ldots d\mu(y_{k+1}) \leq C \prod_{j=1}^{k+1} \|f_j^{p_j - \frac{1}{k+2}}\|_{L^{\sigma_j}(d\mu)}$$

By direct calculation, we have

$$\prod_{j=1}^{k+1} \|f_j^{p_j - \frac{1}{k+2}}\|_{L^{\sigma_j}(d\mu)} = \prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)}^{p_j - \frac{1}{k+2}}.$$

Therefore,

$$\prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)}^{p_j} \leq CA^{\frac{1}{k+2}} \prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)}^{p_j - \frac{1}{k+2}}.$$

That implies

$$\prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)} \leq C \sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \det(y_1, \ldots, y_{k+1})^{\gamma}.$$

□

It should be pointed out that we find condition (i) and (ii) are equivalent to the inequality (2.2.3) in Lemma 2.2.5 as well from the the alternative method of proof (a). Lemma 5 states condition (i) implies inequality (2.2.3), and we use inequality (2.2.3) to get the inequality (2.2.5) in the alternative method of proof (a). Besides, Theorem 2.2.6 shows that condition (i) and (ii) i.e. inequality (2.2.5) are equivalent.

If we strengthen the condition (i) to $\mu(B) \sim |B|_k^{\alpha}$ for all ellipsoids B in H, then $\frac{1}{p_j} < \frac{\gamma}{k\alpha}$ for all $1 \leq j \leq k+1$ and $\frac{\gamma}{\alpha} = \sum_{j=1}^{k+1} \frac{1}{p_j}$ are necessary and sufficient conditions for inequality (2.2.5) to hold, which can be seen in the following theorem.

Theorem 2.2.8. *Let f_j be nonnegative measurable functions defined on real finite-dimensional Hilbert space H. Let μ be a σ-finite, nonnegative Borel measure with satisfying $\mu(B) \sim |B|_k^{\alpha}$ for all ellipsoids B in H. Then for all $0 < p_j < \infty$*

$$\prod_{j=1}^{k+1} \|f_j\|_{L^{p_j}(d\mu)} \leq C_{k,\alpha,p_j} \sup_{y_j} \prod_{j=1}^{k+1} f_j(y_j) \ (\det(y_1, \ldots, y_{k+1}))^{\gamma} \tag{2.2.16}$$

holds, if and only if p_j satisfy

$$\frac{1}{p_j} < \frac{\gamma}{k\alpha} \text{ for all } 1 \leq j \leq k+1 \text{ and } \frac{\gamma}{\alpha} = \sum_{j=1}^{k+1} \frac{1}{p_j}.$$

Proof. $\mu(B) \sim |B|_k^{\alpha}$ for all ellipsoids B in H, so the measure μ is k-curved with exponent α. Theorem 2.2.6 (a) gave the sufficient conditions for inequality (2.2.16) to hold. To see the converse, we study the necessary conditions for inequality (2.2.16) to hold.

Suppose (2.2.16) holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$, then $\frac{\gamma}{\alpha} = \sum_{j=1}^{k+1} \frac{1}{p_j}$ which follows from homogeneity. Let $f_j = \chi_B$ where B is a ball in H,

$j = 1, \dots, k+1$. We consider functions $\chi_B(\frac{\cdot}{R})$ for all $R > 0$: for $j = 1, \dots, k+1$, we have

$$\|\chi_B(\frac{\cdot}{R})\|_{L^{p_j}(d\mu)} \sim R^{\frac{k\alpha}{p_j}} (\mu(B))^{\frac{1}{p_j}},$$

this is because for all $R > 0$

$$\mu(RB) \sim |RB|_k^\alpha = R^{k\alpha} |B|_k^\alpha \sim R^{k\alpha} \mu(B).$$

From the property of $\det(y_1, \dots, y_{k+1})$ it follows that

$$\begin{aligned}
\sup_{y_j} \prod_{j=1}^{k+1} \chi_B(\frac{y_j}{R}) \det(y_1, \dots, y_{k+1})^\gamma &= R^{k\gamma} \sup_{y_j} \prod_{j=1}^{k+1} \chi_B(\frac{y_j}{R}) \det(\frac{y_1}{R}, \dots, \frac{y_{k+1}}{R})^\gamma \\
&= R^{k\gamma} \sup_{y_j} \prod_{j=1}^{k+1} \chi_B(y_j) \det(y_1, \dots, y_{k+1})^\gamma.
\end{aligned}$$

So if (2.2.16) holds, then $\prod_{j=1}^{k+1} R^{\frac{k\alpha}{p_j}} \lesssim R^{k\gamma}$ for all $R > 0$, which implies

$$\sum_{j=1}^{k+1} \frac{k\alpha}{p_j} = k\gamma.$$

That is

$$\frac{\gamma}{\alpha} = \sum_{j=1}^{k+1} \frac{1}{p_j}. \tag{2.2.17}$$

We now claim that if (2.2.16) holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$, p_j must satisfy $\frac{1}{p_j} < \frac{\gamma}{k\alpha}$ for all $1 \le j \le k+1$. Let $f_1 \in L^{p_1}(d\mu)$ be supported on $\{y_1 : |y_1| \ge 10\}$. For $2 \le j \le k+1$, let $f_j = \chi_{B(0,\frac{1}{2})}$ where $B(0, \frac{1}{2})$ denotes the ball in H centred at 0 with radius $\frac{1}{2}$. So $|y_1 - y_j| \sim |y_1|$ for all $2 \le j \le k+1$. We consider the new functions $f_1, f_j(\frac{\cdot}{\epsilon})$ with $0 < \epsilon < 1$, $2 \le j \le k+1$. Suppose that inequality (2.2.16) holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$, then

$$\|f_1\|_{L^{p_1}(d\mu)} \prod_{j=2}^{k+1} \|f_j(\frac{\cdot}{\epsilon})\|_{L^{p_j}(d\mu)} \lesssim \sup_{y_j} \; f_1(y_1) \prod_{j=2}^{k+1} f_j(\frac{y_j}{\epsilon}) \det(y_1, \dots, y_{k+1})^\gamma.$$

By the Hadamard inequality

$$\det(y_1, \dots, y_{k+1}) \le |y_1 - y_{k+1}||y_2 - y_{k+1}| \cdots |y_k - y_{k+1}|,$$

we have

$$\begin{aligned}
&\sup_{y_j} \; f_1(y_1) \textstyle\prod_{j=2}^{k+1} f_j(\frac{y_j}{\epsilon}) \det(y_1, \dots, y_{k+1})^\gamma \\
\le &\sup_{y_j} \; f_1(y_1) \textstyle\prod_{j=2}^{k+1} f_j(\frac{y_j}{\epsilon}) \; (|y_1 - y_{k+1}||y_2 - y_{k+1}| \cdots |y_k - y_{k+1}|)^\gamma.
\end{aligned}$$

So

$$
\begin{aligned}
&\sup_{y_j} \; f_1(y_1)\prod_{j=2}^{k+1} f_j(\frac{y_j}{\epsilon}) \det(y_1,\ldots,y_{k+1})^{\gamma} \\
\leq\; &\sup_{y_j} \; f_1(y_1)\prod_{j=2}^{k+1} f_j(\frac{y_j}{\epsilon}) \; (|y_1-y_{k+1}||y_2-y_{k+1}|\cdots\; |y_k-y_{k+1}|)^{\gamma} \\
\sim \epsilon^{(k-1)\gamma} &\sup_{y_j} \; f_1(y_1)\prod_{j=2}^{k+1} f_j(\frac{y_j}{\epsilon}) \; (|y_1-\frac{y_{k+1}}{\epsilon}||\frac{y_2}{\epsilon}-\frac{y_{k+1}}{\epsilon}|\cdots\; |\frac{y_k}{\epsilon}-\frac{y_{k+1}}{\epsilon}|)^{\gamma} \\
\sim \epsilon^{(k-1)\gamma} &\sup_{y_j} \; \prod_{j=1}^{k+1} f_j(y_j) \; (|y_1-y_{k+1}||y_2-y_{k+1}|\cdots\; |y_k-y_{k+1}|)^{\gamma}.
\end{aligned}
$$

On the other hand, for $2 \leq j \leq k+1$

$$
\|\chi_{B(0,\frac{1}{2})}(\frac{\cdot}{\epsilon})\|_{L^{p_j}(d\mu)} \sim \epsilon^{\frac{k\alpha}{p_j}} \mu(B(0,\frac{1}{2}))^{\frac{1}{p_j}},
$$

this is because for $\epsilon > 0$

$$
\mu(\epsilon B(0,\frac{1}{2})) \sim |\epsilon B(0,\frac{1}{2})|_k^{\alpha} = \epsilon^{k\alpha}|B(0,\frac{1}{2})|_k^{\alpha} \sim \epsilon^{k\alpha}\mu(B(0,\frac{1}{2})).
$$

Then

$$
\|f_1\|_{L^{p_1}(d\mu)}\prod_{j=2}^{k+1}\|f_j(\frac{\cdot}{\epsilon})\|_{L^{p_j}(d\mu)} = \prod_{j=2}^{k+1}\epsilon^{\frac{k\alpha}{p_j}} \prod_{j=1}^{k+1}\|f_j\|_{L^{p_j}(d\mu)}.
$$

So if (2.2.16) holds, then for all $0 < \epsilon < 1$,

$$
\prod_{j=2}^{k+1}\epsilon^{\frac{k\alpha}{p_j}} \lesssim \epsilon^{(k-1)\gamma},
$$

then we have

$$
\sum_{j=2}^{k+1}\frac{k\alpha}{p_j} \geq (k-1)\gamma,
$$

which means

$$
\frac{1}{p_1} = \frac{\gamma}{\alpha} - \sum_{j=2}^{k+1}\frac{1}{p_j} \leq \frac{\gamma}{\alpha} - \frac{k-1}{k\alpha}\gamma = \frac{\gamma}{k\alpha}. \tag{2.2.18}
$$

By symmetry, for any $1 \leq j \leq k+1$ we have $\frac{1}{p_j} \leq \frac{\gamma}{k\alpha}$ provided (2.2.16) holds for all nonnegative functions $f_j \in L^{p_j}(d\mu)$.

As for the boundary case, the following counterexample shows that we must have $\frac{1}{p_j} < \frac{\gamma}{k\alpha}$ for all $1 \leq j \leq k+1$.

For any positive N, let $f_1(y_1) = \frac{1}{|y_1|^{\gamma}}\chi_{2\leq|y_1|\leq N}$, $f_j(y_j) = \chi_{|y_j|\leq 1/4}, 2 \leq j \leq k+1$. The Hadamard inequality tells us

$$
\det(y_1,\ldots,y_{k+1}) \leq |y_1-y_{k+1}|\cdots|y_k-y_{k+1}|,
$$

then

$$\sup_{y_j}\prod_{j=1}^{k+1} f_j(y_j)\det(y_1,\dots,y_{k+1})^\gamma \le \sup_{y_j}\prod_{j=1}^{k+1} f_j(y_j)\ |y_1-y_{k+1}|^\gamma\cdots|y_k-y_{k+1}|^\gamma$$
$$\lesssim \sup_{2\le|y_1|\le N}|y_1|^{-\gamma}(|y_1|+\frac{1}{4})^\gamma \lesssim 1.$$

On the other hand, by polar coordinates we obtain

$$\begin{aligned}\limsup_{N\to\infty}\|f_1\|_{L^{\frac{k\alpha}{\gamma}}(d\mu)} &= \limsup_{N\to\infty}\int_{2\le|y_1|\le N}|y_1|^{-k\alpha}d\mu(y_1)\\ &=\sum_{j>0}\int_{|y_1|\sim 2^j}|y_1|^{-k\alpha}d\mu(y_1)\\ &\gtrsim\sum_{j>0}2^{-k\alpha j}2^{jk\alpha}=\infty,\end{aligned}$$

which gives the contradiction to (2.2.16).
The last inequality follows due to the fact that $\mu(B)\sim|B|_k^\alpha$ for all ellipsoids B in H, which implies

$$\mu(\{y_1\in H:|y_1|\sim 2^j\})\sim 2^{jk\alpha}.$$

□

As is well known, the Lebesgue measure on $\mathbb{R}^n$ is not only n-curved with exponent 1, but also it satisfies $|B|\sim|B|_n$ for all ellipsoids B in H. Hence from Theorem 2.2.8 we obtain the following corollary immediately.

Corollary 2.2.9. *Let f_j be nonnegative measurable functions defined on $L^{p_j}(\mathbb{R}^n)$ with Lebesgue measure. Then*

$$\prod_{j=1}^{n+1}\|f_j\|_{p_j}\le C_{p_j,n}\sup_{y_j}\prod_{j=1}^{n+1} f_j(y_j)\det(y_1,\dots,y_{n+1})^\gamma \tag{2.2.19}$$

holds, if and only if p_j satisfy

$$\tfrac{1}{p_j}<\tfrac{\gamma}{n}\ \ \textit{for all } 1\le j\le n+1\ \textit{ and } \gamma=\sum_{j=1}^{n+1}\frac{1}{p_j}.$$

Additionally, applying the endpoint case of mltilinear determinant inequality Lemma 2.2.7 and Corollary 2.9 for characteristic functions we obtain the following geometric inequalities.

Corollary 2.2.10. *There exists a finite constant C such that for any $y\in\mathbb{R}^n$, for any measurable sets E_j in $\mathbb{R}^n$, $1\le j\le n$,*

$$\prod_{j=1}^{n}|E_j|^{\frac{1}{n}}\le C\sup_{y_1\in E_1,\dots,y_n\in E_n}\det(y,y_1,\dots,y_n), \tag{2.2.20}$$

and

$$\prod_{j=1}^{n+1} |E_j|^{\frac{1}{n+1}} \leq C \sup_{y_1\in E_1,\ldots,y_{n+1}\in E_{n+1}} \det(y_1,\ldots,y_{n+1}). \tag{2.2.21}$$

Let $E_1 = \cdots = E_n = E$, then we have for any measurable set $E \subset \mathbb{R}^n$

$$|E| \leq A_n \sup_{\substack{y_j\in E\\ j=1,\ldots,n}} \det(0,y_1,\ldots,y_n), \tag{2.2.22}$$

$$|E| \leq B_n \sup_{\substack{y_j\in E\\ j=1,\ldots,n+1}} \det(y_1,\ldots,y_{n+1}). \tag{2.2.23}$$

In the introduction we have already discussed the geometric property of (2.2.22)-(2.2.23). The extremal sets of equalities (2.2.20)-(2.2.23) will be discussed in next chapter.

2.2.3 Product Forms of Multilinear Inequalities

We now consider the second class of multilinear inequalities where we have a product form rather than a determinant.

Theorem 2.2.11. *Let $r_{ij} > 0$ and $r_{ij} = r_{ji}$. Let f_j be nonnegative measurable functions defined on $\mathbb{R}^n$, then*

$$\prod_{j=1}^{3} \|f_j\|_{p_j} \leq C_{p_j,r_{ij},n} \sup_{y_j} \prod_{j=1}^{3} f_j(y_j) \prod_{1\leq i<j\leq 3} |y_i - y_j|^{r_{ij}} \tag{2.2.24}$$

holds, if and only if p_j satisfy

$$\sum_{j=1}^{3} \frac{1}{p_j} = \frac{1}{n}(r_{12}+r_{13}+r_{23}) \ , \ \tfrac{1}{p_j} < \tfrac{1}{n}\sum_{i\neq j} r_{ij} \ \ \textit{for every } j.$$

Proof. As before, we start by studying an endpoint version of (2.2.24). Suppose

$$A = \sup_{y_j} f_1(y_1)f_2(y_2)f_3(y_3)|y_1-y_2|^{r_{12}}|y_1-y_3|^{r_{13}}|y_2-y_3|^{r_{23}} < \infty,$$

then there exists measure zero set $E \subset \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, such that

$$f_1(y_1)f_2(y_2)f_3(y_3)|y_1-y_2|^{r_{12}}|y_1-y_3|^{r_{13}}|y_2-y_3|^{r_{23}} \leq A,$$

for all $(y_1,y_2,y_3) \in (\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)\setminus E$. As before, we first turn our attention to the endpoint case of product form inequality (2.2.24). By the definition of $\|f_3\|_\infty$, for any $\varepsilon > 0$ there exists $F \subset \mathbb{R}^n$ such that $|F| > 0$, and for all $y_3 \in F$

$$f_3(y_3) > \|f_3\|_\infty - \varepsilon.$$

So for all $(y_1, ..., y_3) \in (\mathbb{R}^n \times \mathbb{R}^n \times F) \setminus E$,

$$f_2(y_2)(\|f_3\|_\infty - \varepsilon) \leq \frac{1}{|y_1 - y_2|^{r_{12}}|y_2 - y_3|^{r_{23}}} \frac{A}{|y_1 - y_3|^{r_{13}} f_1(y_1)}.$$

Since $|E| = 0$, for almost every $y_3 \in \mathbb{R}^n$

$$|\{(y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n : (y_1, y_2, y_3) \in E\}| = 0.$$

Denote $\{(y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n : (y_1, y_2, y_3) \in E\}$ by G_{y_3}. Because $|F| > 0$, we can choose a $y_3 \in F$ such that $|G_{y_3}| = 0$, which implies for almost every $y_1 \in \mathbb{R}^n$,

$$|\{y_2 \in \mathbb{R}^n : (y_1, y_2) \in G_{y_3}\}| = 0.$$

That means for almost every y_1, almost every y_2

$$(y_1, y_2, y_3) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \setminus E.$$

Thus for almost every y_1, any small $\theta > 0$,

$$\begin{aligned}
&\|f_2\|_{\frac{n}{r_{12}+r_{23}-\theta}}(\|f_3\|_\infty - \varepsilon) \\
&\leq (\int_{\mathbb{R}^n} (|y_1 - y_2|^{-r_{12}}|y_2 - y_3|^{-r_{23}})^{\frac{n}{r_{12}+r_{23}-\theta}} dy_2)^{\frac{r_{12}+r_{23}-\theta}{n}} \frac{A}{|y_1 - y_3|^{r_{13}} f_1(y_1)} \\
&= C(|y_1 - y_3|^{n - \frac{r_{12}n + r_{23}n}{r_{12}+r_{23}-\theta}})^{\frac{r_{12}+r_{23}-\theta}{n}} \frac{A}{|y_1 - y_3|^{r_{13}} f_1(y_1)} \\
&= C|y_1 - y_3|^{-\theta} \frac{A}{|y_1 - y_3|^{r_{13}} f_1(y_1)} \\
&= C\frac{A}{|y_1 - y_3|^{r_{13}+\theta} f_1(y_1)}.
\end{aligned}$$

Take the infimum over y_1, then let $\varepsilon \to 0$,

$$\|f_2\|_{\frac{n}{r_{12}+r_{23}-\theta}}\|f_3\|_\infty \leq C \inf_{y_1} \frac{A}{|y_1 - y_3|^{r_{13}+\theta} f_1(y_1)} = C \frac{A}{\sup\limits_{y_1} |y_1 - y_3|^{r_{13}+\theta} f_1(y_1)}. \tag{2.2.25}$$

In the proof of Lemma 2.1.2, we have stated that for the bilinear form,

$$\|f_1\|_{\frac{n}{r_{13}+\theta},\infty} \lesssim \sup_{y_1} f_1(y_1)|y_1 - y_3|^{r_{13}+\theta}.$$

Therefore, together with (2.2.25) we conclude that for any small $\theta > 0$,

$$\|f_1\|_{\frac{n}{r_{13}+\theta},\infty}\|f_2\|_{\frac{n}{r_{12}+r_{23}-\theta}}\|f_3\|_\infty \lesssim A. \tag{2.2.26}$$

Meanwhile applying the similar arguments we have

$$\|f_1\|_\infty\|f_2\|_{\frac{n}{r_{12}+r_{23}-\theta}}\|f_3\|_{\frac{n}{r_{13}+\theta},\infty} \lesssim A,$$

$$\|f_1\|_{\frac{n}{r_{12}+r_{13}-\theta}}\|f_2\|_\infty\|f_3\|_{\frac{n}{r_{23}+\theta},\infty} \lesssim A,\ \|f_1\|_{\frac{n}{r_{12}+r_{13}-\theta}}\|f_2\|_{\frac{n}{r_{23}+\theta},\infty}\|f_3\|_\infty \lesssim A. \tag{2.2.27}$$

and

$$\|f_1\|_\infty\|f_2\|_{\frac{n}{r_{12}+\theta},\infty}\|f_3\|_{\frac{n}{r_{13}+r_{23}-\theta}} \lesssim A,\ \|f_1\|_{\frac{n}{r_{12}+\theta},\infty}\|f_2\|_\infty\|f_3\|_{\frac{n}{r_{13}+r_{23}-\theta}} \lesssim A. \tag{2.2.28}$$

Since for all $0 < p_j < \infty$, $1 \le j \le 3$ satisfying $\sum_{j=1}^{3}\frac{1}{p_j} = \frac{1}{n}(r_{12}+r_{13}+r_{23})$ and $\frac{1}{p_j} < \frac{1}{n}\sum_{i\neq j} r_{ij}$, we can always find a small θ such that $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$ lies in the interior of the convex hull of $(\frac{r_{13}+\theta}{n}, \frac{r_{12}+r_{23}-\theta}{n}, 0)$, $(\frac{r_{12}+r_{13}-\theta}{n}, 0, \frac{r_{23}+\theta}{n})$, $(0, \frac{r_{12}+\theta}{n}, \frac{r_{13}+r_{23}-\theta}{n})$, $(0, \frac{r_{12}+r_{23}-\theta}{n}, \frac{r_{13}+\theta}{n})$, $(\frac{r_{12}+r_{13}-\theta}{n}, \frac{r_{23}+\theta}{n}, 0)$, $(\frac{r_{12}+\theta}{n}, 0, \frac{r_{13}+r_{23}-\theta}{n})$. By the layer cake representation, we have for $0 < p < q < \infty$

$$\|f\|_q \le C_{p,q}\ \|f\|_{p,\infty}^{\frac{p}{q}}\|f\|_\infty^{1-\frac{p}{q}}.$$

Similar to the discussion in Section 2.1 and Subsection 2.2.2, multilinear product form inequality (2.2.24) follows easily.

Conversely, suppose (2.2.24) holds for all nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^n)$, then

$$\sum_{j=1}^{3}\frac{1}{p_j} = \frac{1}{n}(r_{12}+r_{13}+r_{23})$$

just follows from homogeneity. Besides, by applying the similar example in the proof of Theorem 2.2.8 we can get the necessary conditions for (2.2.24) to hold: for every j

$$\frac{1}{p_j} \le \frac{1}{n}\sum_{i\neq j} r_{ij}.$$

The following counterexample shows that we must have $\frac{1}{p_j} < \frac{1}{n}\sum_{i\neq j} r_{ij}$ for each j. If we assume $\frac{1}{p_1} = \frac{r_{12}+r_{13}}{n}$, for any positive N, let

$$f_1(y_1) = |y_1|^{-(r_{12}+r_{13})}\chi_{2\le|y_1|\le N},\ \ f_2(y_2) = \chi_{|y_2|\le 1/4},\ \ f_3(y_3) = \chi_{|y_3|\le 1/4}.$$

Suppose

$$A = \sup_{y_j} f_1(y_1)f_2(y_2)f_3(y_3)|y_1-y_2|^{r_{12}}|y_1-y_3|^{r_{13}}|y_2-y_3|^{r_{23}},$$

then

$$\begin{aligned} A &\lesssim \sup_{2\le|y_1|\le N} |y_1|^{-(r_{12}+r_{13})}(|y_1|+\frac{1}{4})^{r_{12}}(|y_1|+\frac{1}{4})^{r_{13}} \\ &\lesssim 1. \end{aligned}$$

However, by polar coordinates we obtain

$$\begin{aligned}\|f_1\|_{\frac{n}{r_{12}+r_{13}}} &= \int_{2\le|y_1|\le N} \frac{1}{|y_1|^n} dy_1 \\ &= C\int_2^N \frac{r^{n-1}}{r^n} dr \\ &= C(\ln N - \ln 2) \to \infty,\end{aligned}$$

as $N \to \infty$.

□

Remark 2.2.12. However, our method does not work for multilinear cases for more than three functions. Beckner [3] gave a multilinear fractional integral inequality as follows, mainly applying the general rearrangement inequality (Theorem 3.8 [32]) and the conformally invariant property of (2.2.31) below.

For nonnegative functions $f_j \in L^{p_j}(\mathbb{R}^n)$, $j = 1, \ldots, N$ and $p_j > 1$, $\sum_{j=1}^{N} \frac{1}{p_j} > 1$. Let $0 \le r_{ij} = r_{ji} < n$ be real numbers satisfying

$$\sum_{j=1}^{N} \frac{1}{p'_j} = \frac{1}{n} \sum_{1\le i<j\le N} r_{ij} \qquad (2.2.29),$$

and for every j

$$\frac{1}{p'_j} = \frac{1}{2n} \sum_{i\ne j} r_{ij} \qquad (2.2.30)$$

with p_j and p'_j dual exponents. Then

$$\int_{(\mathbb{R}^n)^N} \prod_{j=1}^{N} f_j(y_j) \prod_{1\le i<j\le N} |y_i - y_j|^{-r_{ij}} dy_1 \ldots dy_N \le C_{p_j,r_{ij},n,N} \prod_{j=1}^{N} \|f_j\|_{p_j} \qquad (2.2.31)$$

Condition (2.2.29) follows from homogeneity. Condition (2.2.30) is to ensure conformal invariance of inequality (2.2.31). Obviously, condition (2.2.30) implies (2.2.29). Similarly to the arguments in the alternative proof of part (a) of Theorem 2.2.6, we have the following theorem.

Theorem 2.2.13. *Let $r_{ij} > 0$ and $r_{ij} = r_{ji}$. Let f_j be nonnegative measurable functions defined on $\mathbb{R}^n$, $1 \le j \le N$. Then*

$$\prod_{j=1}^{N} \|f_j\|_{p_j} \le C_{p_j,r_{ij},n,N} \sup_{y_j} \prod_{j=1}^{N} f_j(y_j) \prod_{1\le i<j\le N} |y_i - y_j|^{r_{ij}}, \qquad (2.2.32)$$

holds for any $0 < p_j < \infty$ satisfying

$$\frac{1}{p_j} = \frac{1}{2n} \sum_{i\ne j} r_{ij}, \quad \sum_{j=1}^{N} \frac{1}{p_j} = \frac{1}{n} \sum_{1\le i<j\le N} r_{ij}. \qquad (2.2.33)$$

Proof. Clearly, the condition $\frac{1}{p_j} = \frac{1}{2n}\sum_{i\neq j} r_{ij}$ satisfies homogeneity condition $\sum_{j=1}^{N}\frac{1}{p_j} = \frac{1}{n}\sum_{1\leq i<j\leq N} r_{ij}$. For any $r_{ij}>0$, denote $\alpha = \sum_{i\neq j} r_{ij}$, then it is easy to see (2.2.32) is equivalent to the following inequality.

$$\prod_{j=1}^{N}\|f_j^{1/\alpha}\|_{p_j\alpha} \leq C_{p_j,r_{ij},n,N}^{1/\alpha} \sup_{y_j}\prod_{j=1}^{N} f_j(y_j)^{1/\alpha}\prod_{1\leq i<j\leq N}|y_i-y_j|^{\frac{r_{ij}}{\alpha}}.$$

Below it is enough to show that

$$\prod_{j=1}^{N}\|f_j\|_{p_j\alpha} \lesssim \sup_{y_j}\prod_{j=1}^{N} f_j(y_j)\prod_{1\leq i<j\leq N}|y_i-y_j|^{\frac{r_{ij}}{\alpha}}. \tag{2.2.34}$$

holds for any $f_j \in L^{p_j\alpha}(\mathbb{R}^n)$ with $0<p_j\alpha<\infty$ satisfying the corresponding conditions

$$\frac{1}{p_j\alpha} = \frac{1}{2n}\sum_{i\neq j}\frac{r_{ij}}{\alpha},\quad \sum_{j=1}^{N}\frac{1}{p_j\alpha} = \frac{1}{n}\sum_{1\leq i<j\leq N}\frac{r_{ij}}{\alpha}. \tag{2.2.35}$$

Suppose $\sup_{y_j}\prod_{j=1}^{N} f_j(y_j)\prod_{1\leq i<j\leq N}|y_i-y_j|^{\frac{r_{ij}}{\alpha}} = A<\infty$. We can write

$$\begin{aligned}
&\|f_1\|_{p_1\alpha}^{p_1\alpha}\cdots\|f_N\|_{p_N\alpha}^{p_N\alpha}\\
&= \int_{(\mathbb{R}^n)^N}\prod_{j=1}^{N} f_j(y_j)^{p_j\alpha}dy_1\dots dy_N\\
&= \int_{(\mathbb{R}^n)^N}\prod_{j=1}^{N} f_j(y_j)\prod_{1\leq i<j\leq N}|y_i-y_j|^{\frac{r_{ij}}{\alpha}}\prod_{j=1}^{N} f_j(y_j)^{p_j\alpha-1}\prod_{1\leq i<j\leq N}|y_i-y_j|^{-\frac{r_{ij}}{\alpha}}dy_1\dots dy_N\\
&\leq A\int_{(\mathbb{R}^n)^N}\prod_{j=1}^{N} f_j(y_j)^{p_j\alpha-1}\prod_{1\leq i<j\leq N}|y_i-y_j|^{-\frac{r_{ij}}{\alpha}}dy_1\dots dy_N.
\end{aligned}$$

Since every $p_j\alpha$ satisfies (2.2.35) and $\sum_{1\leq i<j\leq N}\frac{r_{ij}}{\alpha}=1$, we have $(p_j\alpha)'>1$ and

$$\sum_{j=1}^{N}\frac{1}{(p_j\alpha)'} = N-\sum_{j=1}^{N}\frac{1}{p_j\alpha} = N-\frac{1}{n}>1.$$

Note that from the definition of α, $0<\frac{r_{ij}}{\alpha}<1\leq n$. This allows us to apply inequality (2.2.31) to get

$$\int_{(\mathbb{R}^n)^N}\prod_{j=1}^{N} f_j(y_j)^{p_j\alpha-1}\prod_{1\leq i<j\leq N}|y_i-y_j|^{-\frac{r_{ij}}{\alpha}}dy_1\dots dy_N \lesssim \prod_{j=1}^{N}\|f_j^{p_j\alpha-1}\|_{(p_j\alpha)'}.$$

By direct calculation,

$$\prod_{j=1}^{N} \|f_j^{p_j\alpha-1}\|_{(p_j\alpha)'} = \prod_{j=1}^{N} \|f_j\|_{p_j\alpha}^{p_j\alpha-1}.$$

Combining them together gives

$$\|f_1\|_{p_1\alpha}^{p_1\alpha} \cdots \|f_{k+1}\|_{p_N\alpha}^{p_N\alpha} \lesssim A \prod_{j=1}^{N} \|f_j\|_{p_j\alpha}^{p_j\alpha-1}.$$

This implies that

$$\prod_{j=1}^{N} \|f_j\|_{p_j\alpha} \lesssim A = \sup_{y_j} \prod_{j=1}^{N} f_j(y_j) \prod_{1\le i<j\le N} |y_i - y_j|^{\frac{r_{ij}}{\alpha}},$$

which gives (2.2.34). Therefore by the equivalence as discussed above, this completes the proof of Theorem 2.2.13.

□

Problem 2.2.14. Similarly to the proof of Theorem 2.2.11, it is not hard to see the necessary conditions for inequality (2.2.28) to hold are homogeneity condition and for every j, $1 \le j \le N$,

$$\frac{1}{p_j} < \frac{1}{n} \sum_{i\ne j} r_{ij}.$$

We have already shown that it is sufficient for (3.28) to hold in the trilinear case together with the homogeneity condition. An interesting problem is whether inequality (2.2.28) holds for any p_j satisfying

$$\frac{1}{p_j} < \frac{1}{n} \sum_{i\ne j} r_{ij}, \quad \sum_{j=1}^{N} \frac{1}{p_j} = \frac{1}{n} \sum_{1\le i<j\le N} r_{ij},$$

where $N > 3$.

Chapter 3

Rearrangement Inequalities and Applications

As well known, rearrangement inequalities are useful to find the optimisers of geometric functionals, such as Riesz's inequality, the Pólya-Szegő inequality, the isoperimetric inequality, and Talenti's inequality. In order to find the extremals of Hardy-Littlewood-Sobolev inequality, Riesz's inequality implies that it suffices to maximize over symmetric decreasing functions. Applying the Pólya-Szegő inequality, it suffices to minimize over the symmetric decreasing functions to get the sharp constants of the Sobolev inequality. More applications of classical rearrangement inequalities can be found in [8], [9], [11], [32], and [34]. In this chapter we intoduce some new rearrangement inequalities which will be useful to study the extremals of geometric inequalities discussed in Chapter 2.

3.1 Definition, Notation and Basic Properties

For every nonnegative measurable function f, its layer cake representation is

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)dt,$$

where $\chi_{\{f>t\}}$ is the characteristic function of the level set $\{x : f(x) > t\}$. Let A be a measurable set of finite Lebesgue measure in $\mathbb{R}^n$. The symmetric rearrangement of set A is defined as

$$A^* := \{x : |x| < r\} \equiv B(0, r) \text{ with } |A^*| = |A|.$$

That is, $r^n = \frac{|A|}{v_n}$, and v_n is the volume of unit ball in $\mathbb{R}^n$.

Let f be a nonnegative measurable function that vanishes at infinity, in the sense that its positive level sets have finite measure,

$$|\{x \in \mathbb{R}^n : f(x) > t\}| < \infty, \ \forall\ t > 0.$$

We then define the symmetric decreasing rearrangement of nonnegative measurable function f as

$$\mathcal{R}f(x) \equiv f^*(x) := \int_0^\infty \chi_{\{f>t\}^*}(x)dt, \tag{3.1.1}$$

and define the Steiner symmetrisation of f with respect to the j-th coordinate as

$$\mathcal{R}_j f(x_1, \ldots, x_n) \equiv f^{*j}(x_1, \ldots, x_n) := \int_0^\infty \chi_{\{f(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)>t\}^*}(x_j)dt.$$

We recall another related decreasing rearrangement of f defined on $[0, \infty)$ as

$$f_*(t) = \inf\{\lambda > 0 : m_f(\lambda) \leq t\},$$

where m_f is the distrution function of f,

$$m_f(\lambda) := |\{x \in \mathbb{R}^n : f(x) > \lambda\}|.$$

We observe that $f^*(x) = f_*(v_n|x|^n)$ for any $x \in \mathbb{R}^n$. As is well known, for $0 \leq s, t < \infty$ we have

$$f_*(s) > t \ \text{ if and only if } \ |\{x \in \mathbb{R}^n : f(x) > t\}| > s.$$

By the relation of f^* and f_*, we have for any $s \in \mathbb{R}^n$, $t \geq 0$

$$f^*(s) > t \ \text{ if and only if } \ |\{x \in \mathbb{R}^n : f(x) > t\}| > v_n|s|^n.$$

The rearrangement has many properties (see Lieb-Loss [32], Ch 3). Here we give some properties and propositions which will be used later. It is easy to see that f and f^* are equimeasurable which means

$$|\{x : f(x) > t\}| = |\{x : f^*(x) > t\}|.$$

Together with the layer cake representation of f, the rearrangement is norm preserving, i.e., $\|f\|_p = \|\mathcal{R}f\|_p$ for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Besides, $\|f\|_p = \|\mathcal{R}_n \ldots \mathcal{R}_1 f\|_p$ follows from Fubini's theorem.

Notice that the rearrangement is order preserving. If $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$, then $f^*(x) \leq g^*(x)$ for all $x \in \mathbb{R}^n$. This is mainly because $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$ gives that the level sets of f are contained in the level sets of g. So the level sets of f^* are contained in the level sets of g^*.

Proposition 3.1.1 (Nonexpansivity of rearrangement). *Let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function such that $J(0) = 0$. Let f and g be nonnegative functions on $\mathbb{R}^n$ that vanish at infinity. Then*

$$\int_{\mathbb{R}^n} J(f^*(x) - g^*(x))dx \leq \int_{\mathbb{R}^n} J(f(x) - g(x))dx. \tag{3.1.2}$$

If J is strictly convex, $f = f^*$ and f is strictly decrasing , then equality holds if and only if $g = g^*$.

The detailed proof can be found in Theorem 3.5 of Lieb-Loss [32]. As a result, we have rearrangement is nonexpansive on $L^p(\mathbb{R}^n)$,

$$\|f^* - g^*\|_p \le \|f - g\|_p, \ \forall\ 1 \le p \le \infty.$$

Proposition 3.1.2 (Riesz's rearrangement inequality)**.** *Let f, g, h be nonnegative measuarable functions on $\mathbb{R}^n$ that vanish at infinity. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)h(x-y)dxdy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)g^*(y)h^*(x-y)dxdy. \tag{3.1.3}$$

The proof appears in Theorem 3.7 of Lieb-Loss [32]. This inequality was generalized by Brascamp, Lieb and Luttinger in [7] as follows.

Proposition 3.1.3 (General rearrangement inequality)**.** *Let f_j be nonnegative measuarable functions on $\mathbb{R}^n$ that vanish at infinity, $j = 1, \dots, m$. Let $k \le m$ and let $B = \{b_{ij}\}$ be a $k \times m$ matrix with $1 \le i \le k$, $1 \le j \le m$. Define*

$$I(f_1, \dots, f_m) := \int_{(\mathbb{R}^n)^k} \prod_{j=1}^{m} f_j(\sum_{i=1}^{k} b_{ij}x_i)dx_1 \dots dx_k.$$

Then

$$I(f_1, \dots, f_m) \le I(f_1^*, \dots, f_m^*). \tag{3.1.4}$$

Let $u \in \mathbb{R}^n$ be a unit vector, $u^\perp$ be its orthogonal complement. Then for any $x \in \mathbb{R}^n$, it can be uniquely written as $x = tu + y$ where $y \in u^\perp$. We define the Steiner symmetrisation of A with respect to the direction u as

$$\mathcal{S}_u(A) := \{tu + y : A \cap (\mathbb{R}u + y) \neq \phi, |t| \le \frac{|A \cap (\mathbb{R}u + y)|}{2}\}.$$

Obviously, $\mathcal{R}_j\chi_A$ is the Steiner symmetrisation of A with respect to the direction e_j, $1 \le j \le n$. For simplicity, we denote $\mathcal{S}_{e_n}\mathcal{S}_{e_{n-1}} \dots \mathcal{S}_{e_1}(E)$ by $\mathcal{S}E$, where $\{e_1, \dots, e_n\}$ are the standard orthonormal basis in $\mathbb{R}^n$.

3.2 Rearrangement Inequalities

In this section, we present some new rearrangement inequalities which will be devoted to studying the sharp versions of geometric inequalities.

3.2.1 Bilinear Rearrangement Inequalities

Lemma 3.2.1. *Let E, F be measurable sets of finite volume in $\mathbb{R}^n$. Then*

$$\sup_{x \in E^*, y \in F^*} |x - y| \le \sup_{x \in E, y \in F} |x - y|. \tag{3.2.1}$$

Proof. One can easily check that for any measurable set $C \subset \mathbb{R}^n$ of finite volume,

$$\sup_{x \in C^*} |x| \leq \sup_{x \in C} |x|. \tag{3.2.2}$$

If $\sup_{x \in C} |x| < \sup_{x \in C^*} |x| \equiv s$, there exist positive δ and a measure zero set $M \subset \mathbb{R}^n$ such that $|x| < s - \delta$ for any $x \in C \setminus M$. Then $C \setminus M \subset B(0, s - \delta)$, where $B(0, s - \delta)$ is the ball centred at 0 with radius $s - \delta$. This implies

$$(C \setminus M)^* = C^* \subset B(0, s - \delta),$$

which is a contradiction to $\sup_{x \in C^*} |x| = s$.

It follows from (3.2.2) that

$$\sup_{x \in E, y \in F} |x - y| = \sup_{z \in E - F} |z| \geq \sup_{z \in (E-F)^*} |z|. \tag{3.2.3}$$

The Brunn-Minkowski inequality tells for measurable sets E and F,

$$|E - F|^{1/n} \geq |E|^{1/n} + |F|^{1/n}. \tag{3.2.4}$$

By the definition of symmetric rearrangement of E and F, we have

$$E^* = B(0, r_1), \quad F^* = B(0, r_2),$$

where their radius are $r_1 = (\frac{|E|}{v_n})^{1/n}$, $r_2 = (\frac{|F|}{v_n})^{1/n}$ respectively. Then $E^* + F^*$ is the ball centred at 0 with radius $r_1 + r_2$, and

$$E^* - F^* = E^* + F^* = B(0, r_1 + r_2).$$

Together with the Brunn-Minkowski inequality (3.2.4), we have

$$\begin{aligned} |(E - F)^*|^{1/n} &= |E - F|^{1/n} \geq |E|^{1/n} + |F|^{1/n} \\ &= |E^*|^{1/n} + |F^*|^{1/n} \\ &= v_n^{1/n} r_1 + v_n^{1/n} r_2, \end{aligned}$$

which means

$$|(E - F)^*| \geq v_n (r_1 + r_2)^n = |E^* + F^*|.$$

Therefore

$$E^* - F^* = E^* + F^* \subset (E - F)^*.$$

Applying (3.2.3) gives

$$\sup_{x \in E, y \in F} |x - y| \geq \sup_{z \in (E-F)^*} |z| \geq \sup_{x \in E^*, y \in F^*} |x - y|, \tag{3.2.5}$$

which completes the proof of Lemma 3.2.1.

□

If we consider $E = F$, (3.2.1) and (3.2.2) show that

$$\sup_{x\in E^*} |x| \leq \sup_{x\in E}|x|, \quad \sup_{x,y\in E^*} |x-y| \leq \sup_{x,y\in E} |x-y|. \tag{3.2.6}$$

By the definition of the symmetric rearrangement,

$$E^* = B(0,r), \text{ with } v_n r^n = |E|.$$

Then

$$\sup_{x\in E^*} |x| = r, \quad \sup_{x,y\in E^*} |x-y| = 2r.$$

As discussed in the introduction, we have the following sharp inequalities (3.2.7) and (3.2.8) and both optimisers are balls in $\mathbb{R}^n$. In particular, inequality (3.2.8) is an isodiametric inequality, that is, amongst all sets with given diameter the ball has maximal volume.

Corollary 3.2.2. *Let E be a measurable set of finite volume in $\mathbb{R}^n$. Then*

$$|E| \leq v_n \sup_{x\in E} |x|^n, \tag{3.2.7}$$

$$|E| \leq \frac{v_n}{2^n} \sup_{x,y\in E} |x-y|^n. \tag{3.2.8}$$

Theorem 3.2.3. *Let f, g be nonnegative measurable functions defined on $\mathbb{R}^n$. Then*

$$\sup_{s,t} f^*(s)g^*(t)|s-t| \leq \sup_{x,y} f(x)g(y)|x-y|. \tag{3.2.9}$$

Proof. Suppose $\sup_{x,y} f(x)g(y)|x-y| = A$. We assume for a contradiction that

$$\sup_{s,t} f^*(s)g^*(t)|s-t| > A.$$

Then there exist positive ε and a set $G \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $|G| > 0$ and for all $(s_0, t_0) \in G$ we have

$$f^*(s_0)g^*(t_0)|s_0 - t_0| > A + \varepsilon.$$

It follows from $f^*(s_0) > (A+\varepsilon)(g^*(t_0)|s_0 - t_0|)^{-1}$ and the property of decreasing rearrangement discussed above that

$$|\{x : f(x) > (A+\varepsilon)(g^*(t_0)|s_0 - t_0|)^{-1}\}| > v_n |s_0|^n. \tag{3.2.10}$$

Denote the set $\{x : f(x) > (A+\varepsilon)(g^*(t_0)|s_0 - t_0|)^{-1}\}$ by E, so

$$g^*(t_0) > (A + \frac{\varepsilon}{2}) \, (\inf_{x\in E} f(x)|s_0 - t_0|)^{-1}.$$

Applying the property of decreasing rearrangement again, we have

$$|\{y : g(y) > (A + \frac{\varepsilon}{2})(\inf_{x\in E} f(x)|s_0 - t_0|)^{-1}\}| > v_n |t_0|^n. \tag{3.2.11}$$

Denote the set $\{y : g(y) > (A + \frac{\varepsilon}{2})(\inf_{x\in E} f(x)|s_0 - t_0|)^{-1}\}$ by F. Obviously $s_0 \in E^*, t_0 \in F^*$. Lemma 3.2.1 shows

$$\sup_{x\in E, y\in F} |x - y| \geq \sup_{x\in E^*, y\in F^*} |x - y| \geq |s_0 - t_0|. \tag{3.2.12}$$

Form (3.2.11) it follows that for any $x \in E$, $y \in F$

$$f(x)g(y)|x - y| > (A + \frac{\varepsilon}{2})|s_0 - t_0|^{-1}|x - y|,$$

thus

$$\sup_{x\in E, y\in F} f(x)g(y)|x - y| \geq (A + \frac{\varepsilon}{2})|s_0 - t_0|^{-1} \sup_{x\in E, y\in F} |x - y|.$$

Consequently, together with (3.2.12) we obtain

$$\begin{aligned}\sup_{x,y} f(x)g(y)|x - y| &\geq (A + \frac{\varepsilon}{2})|s_0 - t_0|^{-1} \sup_{x\in E, y\in F} |x - y| \\ &\geq (A + \frac{\varepsilon}{2})|s_0 - t_0|^{-1}|s_0 - t_0| \\ &> A.\end{aligned}$$

This is a contradiction.

□

Remark 3.2.4. By a simple observation, we could replace $|x|$ by any radial increasing function. That is ,

$$\sup_{s,t} f^*(s)g^*(t)h(s - t) \leq \sup_{x,y} f(x)g(y)h(x - y),$$

where h is a radial increasing function on $\mathbb{R}^n$. This is mainly due to the fact that (3.2.1) still holds by replacing $|x|$ by any radial increasing function h:

$$\sup_{x\in E, y\in F} h(x - y) \geq \sup_{x\in E^*, y\in F^*} h(x - y).$$

However, we do not know when there is equality in (3.2.9). One might guess that strict inequality (3.2.9) holds only if $f(x) = f^*(x - y)$ and $g(y) = f^*(x - y)$ for some y in $\mathbb{R}^n$. By the following counterexample, we show that this is not true. In the one-dimensional case, let

$$f(x) = 4\chi_{|x|\leq|E_1|} + \chi_{|E_1|<x\leq|E_1|+2|E_2|}$$

with $|E_1| > |E_2|$, and $f = g$. Then

$$f^*(x) = 4\chi_{|x|\leq|E_1|} + \chi_{|E_1|<|x|\leq|E_1|+|E_2|}.$$

It is easy to check that

$$\sup_{x,y} f(x)g(y)|x - y| = \max\{32|E_1|, 8(|E_1| + |E_2|), 2|E_2|\} = 32|E_1|,$$

and

$$\sup_{x,y} f^*(x)g^*(y)|x-y| = \max\{32|E_1|, 4(2|E_1|+|E_2|), 2(|E_1|+|E_2|)\} = 32|E_1|.$$

So there are other classes of examples where equality holds.

3.2.2 Multilinear Rearrangement Inequalities

Lemma 3.2.4 (the general form of Lemma 3.2.1). *Let $a_j \in \mathbb{R}$ and E_j be sets in $\mathbb{R}$ with finite measure, $j = 1, \ldots, l$. Then*

$$\sup_{x_j \in E_j^*} |\sum_{j=1}^{l} a_j x_j| \le \sup_{x_j \in E_j} |\sum_{j=1}^{l} a_j x_j|. \tag{3.2.13}$$

Proof.

From the Brunn-Minkowski inequality

$$|E + F| \ge |E| + |F|$$

where $E, F \subset \mathbb{R}$, it follows that

$$|E_1 + \cdots + E_l| \ge |E_1| + \cdots + |E_l|.$$

Because $E_j^* = (-|E_j|/2, |E_j|/2)$, $1 \le j \le l$, then

$$E_1^* + \cdots + E_l^* = (-\sum_{j=1}^{l} \frac{|E_j|}{2}, \sum_{j=1}^{l} \frac{|A_j|}{2}).$$

Thus we have

$$|(E_1 + \cdots + E_l)^*| = |E_1 + \cdots + E_l| \ge |E_1| + \cdots + |E_l| = |E_1^* + \cdots + E_l^*|,$$

which implies

$$(E_1 + \cdots + E_l)^* \supset E_1^* + \cdots + E_l^*. \tag{3.2.14}$$

Clearly, for any non-zero $a \in \mathbb{R}$ and any measurable subset E in $\mathbb{R}$

$$(aE)^* = aE^*. \tag{3.2.15}$$

Combining with (3.2.14)-(3.2.15) we have

$$(a_1 E_1 + \cdots + a_l E_l)^* \supset a_1 E_1^* + \cdots + a_l E_l^*. \tag{3.2.16}$$

Apply (3.2.2) and (3.2.16),

$$\sup_{x_j \in E_j} |\sum_{j=1}^{l} a_j x_j| = \sup_{\bar{x} \in \sum_{j=1}^{l} a_j E_j} |\bar{x}| \ge \sup_{\bar{x} \in (\sum_{j=1}^{l} a_j E_j)^*} |\bar{x}| \ge \sup_{\bar{x} \in \sum_{j=1}^{l} a_j E_j^*} |\bar{x}|.$$

Besides,

$$\sup_{\bar{x}\in\sum_{j=1}^{l} a_jE_j^*} |\bar{x}| = \sup_{x_j\in a_jE_j^*} |\sum_{j=1}^{l} x_j| = \sup_{x_j\in E_j^*} |\sum_{j=1}^{l} a_jx_j|.$$

Therefore,

$$\sup_{x_j\in E_j} |\sum_{j=1}^{l} a_jx_j| \geq \sup_{x\in E_j^*} |\sum_{j=1}^{l} a_jx_j|.$$

$\square$

Theorem 3.2.5 (the general form of Theorem 3.2.3)**.** *Let f_j be nonnegative measurable functions defined on $\mathbb{R}$ and let a_j be real numbers, then*

$$\sup_{x_j}\prod_{j=1}^{l} f_j^*(x_j)|\sum_{j}^{l} a_jx_j| \leq \sup_{x_j}\prod_{j=1}^{l} f_j(x_j)|\sum_{j}^{l} a_jx_j|. \tag{3.2.17}$$

Proof. Suppose $\sup_{x_j}\prod_{j=1}^{l} f_j(x_j)|\sum_{j}^{l} a_jx_j| = A < \infty$. We assume for a contradiction that

$$\sup_{x_j}\prod_{j=1}^{l} f_j^*(x_j)|\sum_{j}^{l} a_jx_j| > A.$$

Then there exist positive ε and a set $G \subset \mathbb{R}\times\cdots\times\mathbb{R}$ such that $|G| > 0$ and for all $(z_1,\ldots,z_l) \in G$ we have

$$\prod_{j=1}^{l} f_j^*(z_j)|\sum_{j}^{l} a_jz_j| > A+\varepsilon. \tag{3.2.18}$$

So

$$f_1^*(z_1) > (A+\varepsilon)(\prod_{j=2}^{l} f_j^*(z_j)|\sum_{j}^{l} a_jz_j|)^{-1}. \tag{3.2.19}$$

Define the set

$$E_1 := \{x_1 : f_1(x_1) > (A+\varepsilon)(\prod_{j=2}^{l} f_j^*(z_j)|\sum_{j}^{l} a_jz_j|)^{-1}\},$$

so by the property of decreasing rearrangement together with (3.2.19)

$$|E_1| > 2|z_1|.$$

From the definition of E_1

$$f_2^*(z_2) > (A+\frac{\varepsilon}{2})\ (\inf_{x_1\in E_1} f_1(x_1)\prod_{j=3}^{l} f_j^*(z_j)|\sum_{j}^{l} a_jz_j|)^{-1}.$$

We then define

$$E_2 = \{x_2 : f_2(x_2) > (A + \frac{\varepsilon}{2})(\inf_{x_1 \in E_1} f_1(x_1) \prod_{j=3}^{l} f_j^*(z_j) |\sum_j^l a_j z_j|)^{-1}\},$$

so

$$|E_2| > 2|z_2|.$$

Overall, we can take the similar arguments to define sets E_k, $1 < k < l$

$$E_k = \{x_k : f_k(x_k) > (A + \frac{\varepsilon}{k})(\prod_{j=1}^{k-1} \inf_{x_j \in E_j} f_j(x_j) \prod_{j=k+1}^{l} f_j^*(z_j) |\sum_j^l a_j z_j|)^{-1}\},$$

and

$$E_l = \{x_l : f_l(x_l) > (A + \frac{\varepsilon}{l})(\prod_{j=1}^{l-1} \inf_{x_j \in E_j} f_j(x_j) |\sum_j^l a_j z_j|)^{-1}\}. \tag{3.2.20}$$

It is easily seen that for each $j = 1, \ldots, l$

$$|E_j| > 2|z_j|, \tag{3.2.21}$$

and thus $z_j \in E_j^*$.

Applying Lemma 3.2.4 gives

$$\sup_{x_j \in E_j^*} |\sum_{j=1}^{l} a_j x_j| \leq \sup_{x_j \in E_j} |\sum_{j=1}^{l} a_j x_j|.$$

Since $z_j \in E_j^*$, $j = 1, \ldots, l$,

$$|\sum_j^l a_j z_j| \leq \sup_{x_j \in E_j} |\sum_{j=1}^{l} a_j x_j|. \tag{3.2.22}$$

From the definition of E_l in (3.2.20) we have for any $x_j \in E_j$, $1 \leq j \leq l$

$$\prod_{j=1}^{l} f_j(x_j) |\sum_j^l a_j x_j| > (A + \frac{\varepsilon}{l}) |\sum_j^l a_j z_j|^{-1} |\sum_j^l a_j x_j|.$$

Therefore, together with (3.2.22) we obtain

$$A \geq \sup_{x_j \in E_j} \prod_{j=1}^{l} f_j(x_j) \; |\sum_{j=1}^{l} a_j x_j| > (A + \frac{\varepsilon}{l}) |\sum_j^l a_j z_j|^{-1} \sup_{x_j \in E_j} |\sum_j^l a_j x_j| > A,$$

which gives a contradiction. This completes the proof of Theorem 3.2.5.

$\square$

It follows from Theorem 3.2.5 that we have the following corollary.

Corollary 3.2.6. *Let f_j be defined on $\mathbb{R}^n$, $j = 1, \dots, n+1$. Then for any $1 \le i \le n$,*

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^{*i}(y_j) \det(y_1, \dots, y_{n+1}) \le \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \dots, y_{n+1}). \tag{3.2.23}$$

Furthermore,

$$\sup_{y_j} \prod_{j=1}^{n+1} \mathcal{R}_n \mathcal{R}_{n-1} \dots \mathcal{R}_1 f_j(y_j) \det(y_1, \dots, y_{n+1}) \le \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \dots, y_{n+1}).$$

This is because for each i, $1 \le i \le n$, $\det(y_1, \dots, y_{n+1})$ is the linear combination of $y_{1i}, \dots, y_{(n+1)i}$, where y_{ki} is the i-th coordinate of $y_k \in \mathbb{R}^n, 1 \le k \le n+1$. See the proof of Lemma 3.3.1 for the details. In the next section we will improve inequality (3.2.23) as shown in Theorem 3.3.4.

3.3 Some Applications

This section deals with some applications of rearrangement inequalities above. We mainly discuss two kinds of inequalities related to isodiametric problems: the geometric determinant inequalities in Corollary 2.2.10 and the matrix inequalities discussed in Section 1.3. Clearly, the inequalities (2.2.20)-(2.2.23) in Corollary 2.2.10 are the determinant form of multilinear inequalities for characteristic funcitons. In Subsection 3.3.1 we obtain the extremal sets of inequalities (2.2.20)-(2.2.23) combining with the rearrangement tools. Besides, we find those rearrangement theorem above will be essential to deduce the matrix inequalities in Subsection 3.3.2. However, the sharp versions of matrix inequalities have not been determined. It is still an open probelm.

3.3.1 Applications in Determinant Inequalities

As an application of multilinear rearrangement inequalities above, we obtain Theorem 3.3.2 which are helpful to determine the extremal sets of inequalities (2.2.20)-(2.2.23). It follows from Lemma 3.2.4 we have the following lemma.

Lemma 3.3.1. *Let E_j be measurable sets in $\mathbb{R}^n$ $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$, then for each $1 \le i \le n$,*

$$\sup_{\substack{y_j \in \mathcal{S}_{e_i}(E_j) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n), \tag{3.3.1}$$

and

$$\sup_{\substack{y_j \in \mathcal{S}_{e_i}(E_j) \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}). \tag{3.3.2}$$

Proof. For simplicity, we just see (3.3.1)-(3.3.2) hold for e_1. Define the projection π: $\mathbb{R}^n \to \mathbb{R}^{n-1}$ by

$$\pi(x) = (x_2, \dots, x_n),\ \forall\ x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

For any $x \in \mathbb{R}^n$, write $x = (x_1, x')$ where $x' \in \mathbb{R}^{n-1}$. For $y_j \in E_j$,

$$\det(0, y_1, \dots, y_n) = |\det \begin{pmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ \vdots & \vdots & & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn} \end{pmatrix}| = |y_{11}A_1 + y_{21}A_2 + \dots y_{n1}A_n|,$$

where A_j depend only on $\{y'_1, \dots, y'_n\}$. For each j, fix $y'_j := (y_{j2}, \dots, y_{jn}) \in \pi(E_j)$. Let

$$E_j(y'_j) = \{y_{j1} \in \mathbb{R} : (y_{j1}, y'_j) \in E_j\}.$$

It follows from Lemma 3.2.4 that

$$\sup_{y_{j1} \in E_j(y'_j)^*} |\sum_{j=1}^n A_j y_{j1}| \le \sup_{y_{j1} \in E_j(y'_j)} |\sum_{j=1}^n A_j y_{j1}|. \tag{3.3.3}$$

Since

$$\mathcal{S}_{e_1}(E_j) = \bigcup_{y'_j \in \pi(E_j)} \{(y_{j1}, y'_j) : y_{j1} \in E_j(y'_j)^*\}, \tag{3.3.4}$$

together with (3.3.3) gives

$$\sup_{\substack{y_j \in \mathcal{S}_{e_1}(E_j) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n).$$

Similarly, because

$$\begin{aligned} \det(y_1, \dots, y_{n+1}) &= |\det \begin{pmatrix} y_{11} - y_{(n+1)1} & y_{21} - y_{(n+1)1} & \cdots & y_{n1} - y_{(n+1)1} \\ \vdots & \vdots & & \vdots \\ y_{1n} - y_{(n+1)n} & y_{2n} - y_{(n+1)n} & \cdots & y_{nn} - y_{(n+1)n} \end{pmatrix}| \\ &= |y_{11}A_1 + y_{21}A_2 + \cdots + y_{(n+1)1}A_{n+1}|, \end{aligned}$$

where A_j depend only on $\{y'_1, \dots, y'_{n+1}\}$. By similar arguments we can obtain

$$\sup_{\substack{y_j \in \mathcal{S}_{e_1}(E_j) \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}).$$

$\square$

Generally, together with the rotation invariance we have the following rearrangement theorem.

Theorem 3.3.2. *Let E_j be measurable set in $\mathbb{R}^n$, $1 \le j \le n$. Let u be a unit* *vector in* $\mathbb{R}^n$. *Then*

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n). \tag{3.3.5}$$

and

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j) \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}). \tag{3.3.6}$$

Proof. Suppose $u = \rho e_i$, where ρ is a rotation around the origin in $\mathbb{R}^n$.
By definition,

$$\begin{aligned}
\mathcal{S}_{\rho e_i}(E) &= \{t\rho e_i + y : E \cap [\mathbb{R}(\rho e_i) + y] \neq \phi, |t| \le \frac{|E \cap [\mathbb{R}(\rho e_i) + y]|}{2}\} \\
&= \{\rho(te_i + \rho^{-1}y) : \rho^{-1}(E) \cap (\mathbb{R}e_i + \rho^{-1}y) \neq \phi, |t| \le \frac{|\rho[\rho^{-1}(E) \cap (\mathbb{R}e_i + \rho^{-1}y)]|}{2}\} \\
&= \{\rho(te_i + \rho^{-1}y) : \rho^{-1}(E) \cap (\mathbb{R}e_i + \rho^{-1}y) \neq \phi, |t| \le \frac{|\rho^{-1}(E) \cap (\mathbb{R}e_i + \rho^{-1}y)|}{2}\}.
\end{aligned}$$

Note that

$$\mathcal{S}_{e_i}(\rho^{-1}(E)) = \{te_i + \rho^{-1}y : \rho^{-1}(E) \cap (\mathbb{R}e_i + \rho^{-1}y) \neq \phi, |t| \le \frac{|\rho^{-1}(E) \cap (\mathbb{R}e_i + \rho^{-1}y)|}{2}\}.$$

Hence we obtain

$$\mathcal{S}_{\rho e_i}(E) = \rho \circ \mathcal{S}_{e_i}(\rho^{-1}(E)). \tag{3.3.7}$$

Since rotation ρ does not change the right side of (3.3.5) and (3.3.6),

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) = \sup_{\substack{y_j \in \rho \circ \mathcal{S}_{e_i}(\rho^{-1}(E_j)) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) = \sup_{\substack{y_j \in \mathcal{S}_{e_i}(\rho^{-1}(E_j)) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n).$$

Applying (3.3.1) gives

$$\sup_{\substack{y_j \in \mathcal{S}_{e_i}(\rho^{-1}(E_j)) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) \le \sup_{\substack{y_j \in \rho^{-1}(E_j) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) = \sup_{\substack{y_j \in E_j \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n).$$

Therefore, we conclude

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j) \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n}} \det(0, y_1, \dots, y_n).$$

Likewise, applying (3.3.2)

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j) \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}) \le \sup_{\substack{y_j \in E_j \\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}).$$

□

Now we can decide the sharp versions of the determinant inqualities in this section. It is known that, given a compact convex set $K \subset \mathbb{R}^n$, there exists a sequence iterated Steiner symmetrisation of K that converges in the Hausdorff metric to a ball of the same volume. For example, given a basis of unit directions $u_1, \ldots, u_n$ for $\mathbb{R}^n$ having mutually irrational multiple of π radian differences, then the sequence $\mathcal{S}_{u_n} \ldots \mathcal{S}_{u_2} \mathcal{S}_{u_1}(K)$ iterated infinitely many times to K will converge to a ball of the same volume as K. For the convergence of Steiner symmetrisation, refer to [4], [6], [18], [30], [48], etc.

One can easily verify that the suprema function on the right side of inequalities (3.3.1)-(3.3.2) are continuous under the Hausdorff metric, and they do not change if we replace each E_j by $\overline{\mathrm{co}}(E_j)$. Therefore, applying the convergence of Steiner symmetrisation together with Theorem 3.3.2 we have shown the following lemma.

Lemma 3.3.3. *For any set $E_j \subset \mathbb{R}^n$ of finite measure, $1 \le j \le n$,*

$$\sup_{y_1 \in E_1^*, \ldots, y_n \in E_n^*} \det(0, y_1, \ldots, y_n) \le \sup_{y_1 \in E_1, \ldots, y_n \in E_n} \det(0, y_1, \ldots, y_n), \tag{3.3.8}$$

and

$$\sup_{y_1 \in E_1^*, \ldots, y_{n+1} \in E_{n+1}^*} \det(y_1, \ldots, y_{n+1}) \le \sup_{y_1 \in E_1, \ldots, y_{n+1} \in E_{n+1}} \det(y_1, \ldots, y_{n+1}). \tag{3.3.9}$$

Combining with Lemma 3.3.3 we obtain the multilinear functional rearrangement inequalities.

Theorem 3.3.4. *Let f_j be nonnegative measurable functions vanishing at infinity on $\mathbb{R}^n$. Then we have*

$$\sup_{y_j} \prod_{j=1}^{n} f_j^*(y_j) \det(0, y_1, \ldots, y_n) \le \sup_{y_j} \prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \ldots, y_n), \tag{3.3.10}$$

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^*(y_j) \det(y_1, \ldots, y_{n+1}) \le \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1}), \tag{3.3.11}$$

where the sup *is the essential supremum.*

Proof. Suppose $\sup_{y_j} \prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \ldots, y_n) = A < \infty$. We assume for a contradiction that

$$\sup_{y_j} \prod_{j=1}^{n} f_j^*(y_j) \det(0, y_1, \ldots, y_n) > A.$$

Then there exist positive ε and a set $G \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ such that $|G| > 0$ and for all $(x_1, \ldots, x_n) \in G$ we have

$$\prod_{j=1}^{n} f_j^*(x_j) \det(0, x_1, \ldots, x_n) > A + \varepsilon, \tag{3.3.12}$$

which gives

$$f_1^*(x_1) > (A+\varepsilon)(\prod_{j=2}^{n} f_j^*(x_j) \det(0, x_1, \ldots, x_n))^{-1}. \tag{3.3.13}$$

Define the set

$$E_1 := \{y_1 : f_1(y_1) > (A+\varepsilon)(\prod_{j=2}^{n} f_j^*(x_j) \det(0, x_1, \ldots, x_n))^{-1}\},$$

so by the property of decreasing rearrangement together with (3.3.13)

$$|E_1| > v_n |x_1|^n.$$

From the definition of E_1

$$f_2^*(x_2) > (A+\frac{\varepsilon}{2})\ (\inf_{y_1 \in E_1} f_1(y_1) \prod_{j=3}^{n} f_j^*(x_j) \det(0, x_1, \ldots, x_n))^{-1}.$$

We then define

$$E_2 = \{y_2 : f_2(y_2) > (A+\frac{\varepsilon}{2})(\inf_{y_1 \in E_1} f_1(y_1) \prod_{j=3}^{n} f_j^*(x_j) \det(0, x_1, \ldots, x_n))^{-1}\}.$$

So

$$|E_2| > v_n |x_2|^n.$$

Overall, we can take the similar arguments to define sets E_k, $1 < k < n$

$$E_k = \{y_k : f_k(y_k) > (A+\frac{\varepsilon}{k})(\prod_{j=1}^{k-1} \inf_{y_j \in E_j} f_j(y_j) \prod_{j=k+1}^{n} f_j^*(x_j) \det(0, x_1, \ldots, x_n))^{-1}\},$$

and

$$E_n = \{y_n : f_n(y_n) > (A+\frac{\varepsilon}{n})(\prod_{j=1}^{n-1} \inf_{y_j \in E_j} f_j(y_j) \det(0, x_1, \ldots, x_n))^{-1}\}.$$

It is easily seen that for each $j = 1, \ldots, n$

$$|E_j| > v_n |x_j|^n, \tag{3.3.14}$$

and thus $x_j \in E_j^*$. It follows from Lemma 3.3.3 that

$$\sup_{y_1 \in E_1^*, \ldots, y_n \in E_n^*} \det(0, y_1, \ldots, y_n) \le \sup_{y_1 \in E_1, \ldots, y_n \in E_n} \det(0, y_1, \ldots, y_n).$$

That together with $x_j \in E_j^*$, $k = 1, \ldots, n$, implies

$$\det(0, x_1, \ldots, x_n) \le \sup_{y_1 \in E_1, \ldots, y_n \in E_n} \det(0, y_1, \ldots, y_n). \tag{3.3.15}$$

From the definition of E_n we have for any $y_j \in E_j$, $1 \le j \le n$

$$\prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \dots, y_n) > (A + \frac{\varepsilon}{n})(\det(0, x_1, \dots, x_n))^{-1} \det(0, y_1, \dots, y_n).$$

Therefore, together with (3.3.15) we obtain

$$\begin{aligned} A &\ge \sup_{y_1 \in E_1, \dots, y_n \in E_n} \prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \dots, y_n) \\ &> (A + \frac{\varepsilon}{n})(\det(0, x_1, \dots, x_n))^{-1} \sup_{y_1 \in E_1, \dots, y_n \in E_n} \det(0, y_1, \dots, y_n) > A. \end{aligned}$$

This gives a contradiction. Taking the similar arguments combining with (3.3.9) concludes that (3.3.11) holds.

□

Let $f_j = \chi_{E_j}$, and E_j be measurable set in $\mathbb{R}^n$. Applying Theorem 3.3.4 we obtain the following two sharp "multilinear" determinant inequalties suggested by the multilinear perspective of (2.2.20).

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n}} \le A_n \sup_{y_1 \in E_1, \dots, y_n \in E_n} \det(0, y_1, \dots, y_n), \tag{3.3.16}$$

and

$$\prod_{j=1}^{n+1} |E_j|^{\frac{1}{n+1}} \le B_n \sup_{y_1 \in E_1, \dots, y_{n+1} \in E_{n+1}} \det(y_1, \dots, y_{n+1}). \tag{3.3.17}$$

Moreover, they are both extremised by balls centred at 0. By the Hadamard inequality, it is easy to calculate $\sup_{y_1 \in E_1, \dots, y_n \in E_n} \det(0, y_1, \dots, y_n)$ when E_j are balls centred at 0, and thus to get the sharp constants A_n. However, the sharp constant B_n of (3.3.17) has not been calculated in this thesis. It follows from Theorem 3.3.4 that we also obtain the optimisers for (1.3.7) and (1.3.8) which is the special case when $f_j = \chi_E$.

3.3.2 Applications in Matrix Inequalities

In this subsection we improve the matrix inequalities in Sublemma 14.1 and Lemma 13.2 discussed in the introduction. All the proof mainy rely on the rearrangement inequality in Lemma 3.2.4 and an invariance under action of $O(n)$ by premultiplication as described in the introduction.

Theorem 3.3.5. *There exists a finite constant $\mathcal{C}_n$ such that for any measurable set $E_j \subset \mathfrak{M}^{n \times n}$ of finite measure, $j = 1, \dots, n$,*

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n^2}} \le \mathcal{C}_n \sup_{\substack{A_j \in E_j \\ j=1,\dots,n}} |\det(A_1 + \dots + A_n)|, \tag{3.3.18}$$

where $|\cdot|$ denotes the Lebesgue measure on Euclidean space $\mathbb{R}^{n^2}$ and the absolute value on $\mathbb{R}$.

Proof. Suppose

$$\sup_{\substack{A_j \in E_j \\ j=1,\dots,n}} |\det(A_1 + \cdots + A_n)| = s < \infty.$$

First we give some definition and notation. Let $F \subset \mathfrak{M}^{n\times m}$, define

$$v(F) = \{ \begin{pmatrix} a_{11} & a_{21} & \dots & a_{(m-1)1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{(m-1)n} \end{pmatrix} : \exists \begin{pmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix} \text{ such that } \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix} \in F \},$$

so $v(F) \subset \mathfrak{M}^{n\times(m-1)}$. For any n-by-$(m-1)$ matrix

$$x = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{(m-1)1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{(m-1)n} \end{pmatrix} \in v(F),$$

we denote

$$F^x = \{ \begin{pmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix} : \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix} \in F \} \subset \mathfrak{M}^{n\times 1}.$$

Let $E \subset \mathfrak{M}^{n\times n}$. For any rotation around the origin T in $\mathbb{R}^n$, consider

$$\Phi_T : A \mapsto TA, \ \forall \ A \in E,$$

where T is a n-by-n matrix with $\det(T) = 1$. Note that Φ_T does not change $|E|$ and $\sup_{A\in E} |\det(A)|$. This is because

$$\sup_{A\in\Phi_T(E)} |\det(A)| = \sup_{A\in E} |\det(TA)| = \sup_{A\in E} |\det(A)|. \tag{3.3.19}$$

Besides, if we see the matrix $A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \in E$ as a vector

$$(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}) \in \mathbb{R}^{n^2},$$

then the matrix $\Phi_T(A)$ becomes

$$\begin{pmatrix} T & & & \\ & T & & \\ & & \ddots & \\ & & & T \end{pmatrix} \begin{pmatrix} a_{11} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Thus
$$|\Phi_T(E)| = |T|^n|E| = |E|. \tag{3.3.20}$$
From $|E| = \int_{v(E)} |E^x|dx$ it follows that there always exists $\overline{x} \in v(E)$ such that
$$|v(E)||E^{\overline{x}}| \gtrsim_n |E|. \tag{3.3.21}$$

By John Ellipsoid, for any compact convex $G \subset \mathbb{R}^n$ there exists an ellipsoid $G' \subset G$ such that
$$|G'| \gtrsim_n |G|. \tag{3.3.22}$$
For the John ellipsoid G', we choose a rotation $T \in O(n)$ such that TG' is an ellipsoid with principal axes parallel to the coordinate axes. As well known, for every ellipsoid TG' with principal axes parallel to the coordinate axes, there exists an axis-parallel rectangle $H \subset TG'$ such that
$$|H| \gtrsim_n |TG'|. \tag{3.3.23}$$
Hence if $E^{\overline{x}}$ is convex, from (3.3.22)-(3.3.23) we may assume that there exists $T \in O(n)$ such that $E^{\overline{x}}$ is an axis-parallel rectangle in $\mathbb{R}^n$.

Take $n = 2$. By (3.3.21) there exists
$$x_{10} \in v(E_1) \subset \mathfrak{M}^{2\times 1}, \quad x_{20} \in v(E_2) \subset \mathfrak{M}^{2\times 1}$$
such that
$$|v(E_1)||E_1^{x_{10}}| \gtrsim |E_1|, \quad |v(E_2)||E_2^{x_{20}}| \gtrsim |E_2|. \tag{3.3.24}$$
Then
$$\max\{|v(E_2)||E_1^{x_{10}}|, |v(E_1)||E_2^{x_{20}}|\} \gtrsim (|E_1||E_2|)^{1/2}.$$
For simplicity, suppose
$$|v(E_2)||E_1^{x_{10}}| \gtrsim (|E_1||E_2|)^{1/2}. \tag{3.3.25}$$

To study the suprema, we consider 2-by-2 matrix
$$\overline{A}_1 := \begin{pmatrix} (x_{10})_1 & (x_{10})_2 \end{pmatrix} \in E_1$$
with $(x_{10})_1 = x_{10} \in \mathfrak{M}^{n\times 1}$ and $(x_{10})_2 \in E_1^{x_{10}}$.
For any $\overline{A}_2 := \begin{pmatrix} x_1 & x_2 \end{pmatrix} \in E_2$, for any constructed $\overline{A}_1$ above
$$\begin{aligned} s &\geq |\det(\overline{A}_1 + \overline{A}_2)| \\ &= |\det\begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}|. \end{aligned}$$
So fix the first column, we have for any $x_1 \in v(E_2)$, $x_2 \in E_2^{x_1}$
$$s \geq \sup_{(x_{10})_2 \in E_1^{x_{10}}} |\det\begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}|. \tag{3.3.26}$$

Because fix all the columns except one, the $|\det|$ function is convex function of the remaining column. Thus

$$s \geq \sup_{(x_{10})_2 \in \mathrm{co} E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}|. \tag{3.3.27}$$

By (3.3.22) we may assume $\mathrm{co}E_1^{x_{10}}$ is an ellipsoid in $\mathbb{R}^2$. Choose a rotation $T_0 \in O(2)$ such that $T_0 \mathrm{co} E_1^{x_{10}}$ is an ellipsoid with principal axes parallel to the coordinate axes. From (3.3.23) there exists an axis-parallel rectangle $L_{T_0 \mathrm{co} E_1^{x_{10}}} \subset T_0 \mathrm{co} E_1^{x_{10}}$ such that

$$|L_{T_0 \mathrm{co} E_1^{x_{10}}}| \geq C |T_0 \mathrm{co} E_1^{x_{10}}|.$$

Note that (3.3.27) is invariant under $O(2)$ as discussed in (3.3.19), so

$$\begin{aligned} s &\geq \sup_{(x_{10})_2 \in \mathrm{co} E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}| \\ &= \sup_{(x_{10})_2 \in \mathrm{co} E_1^{x_{10}}} |\det \begin{pmatrix} T_0 x_1 + T_0(x_{10})_1 & T_0 x_2 + T_0 (x_{10})_2 \end{pmatrix}|. \end{aligned} \tag{3.3.28}$$

Since $L_{T_0 \mathrm{co} E_1^{x_{10}}}$ is an axis-parallel rectangle in $\mathbb{R}^2$, it can be written as $A_1 \times A_2$, where A_1, A_2 are intervals in $\mathbb{R}$, and then

$$\mathcal{S}(L_{T_0 \mathrm{co} E_1^{x_{10}}}) = \mathcal{S}(L_{T_0 \mathrm{co} E_1^{x_{10}}} + T_0 x_2) = A_1^* \times A_2^*, \ \forall \ x_2 \in E_2^{x_1}.$$

Similar to the proof of Lemma 3.3.1, applying Lemma 3.2.4 gives for any $x_1 \in v(E_2)$

$$s \geq \sup_{(x_{10})_2 \in \mathcal{S}(L_{T_0 \mathrm{co} E_1^{x_{10}}})} |\det \begin{pmatrix} T_0 x_1 + T_0 (x_{10})_1 & (x_{10})_2 \end{pmatrix}|. \tag{3.3.29}$$

Therefore, by (2.2.20) we deduce that

$$s \geq C |T_0 v(E_2) + T_0 (x_{10})_1|^{1/2} |\mathcal{S}(L_{T_0 \mathrm{co} E_1^{x_{10}}})|^{1/2} \geq C |v(E_2)|^{1/2} |\mathrm{co} E_1^{x_{10}}|^{1/2}.$$

This together with (3.3.25) implies

$$s \geq C |v(E_2)|^{1/2} |\mathrm{co} E_1^{x_{10}}|^{1/2} \geq C |v(E_2)|^{1/2} |E_1^{x_{10}}|^{1/2} \geq C (|E_1||E_2|)^{1/4},$$

which completes (3.3.18) for $n = 2$.

Take $n = 3$. By (3.3.21) for each E_j there exists $x_{j0} \in v(E_j) \subset \mathfrak{M}^{3\times 2}$ such that

$$|v(E_j)||E_j^{x_{j0}}| \gtrsim |E_j|, 1 \leq j \leq 3. \tag{3.3.30}$$

Denote $F_j = v(E_j) \subset \mathfrak{M}^{3\times 2}$, there exists fixed $x_{j1} \in v(F_j) \subset \mathfrak{M}^{3\times 1}$ such that

$$|v(F_j)||F_j^{x_{j1}}| \gtrsim |F_j| = v(E_j). \tag{3.3.31}$$

From (3.3.30)-(3.3.31), we have $x_{j0} \in v(E_j), x_{j1} \in v(F_j)$

$$|v(F_j)||F_j^{x_{j1}}||E_j^{x_{j0}}| \gtrsim |E_j|, 1 \leq j \leq 3. \tag{3.3.32}$$

It is not hard to see there exists $\{i_1, i_2, i_3\}$ with $i_1 \neq i_2 \neq i_3$ such that

$$(v(F_{i_3})||F_{i_2}^{x_{i_2 1}}||E_{i_1}^{x_{i_1 0}}|)^3 \geq \prod_{j=1}^{3}(|v(F_j)||F_j^{x_{j1}}||E_j^{x_{j0}}|) \gtrsim \prod_{j=1}^{3}|E_j|. \tag{3.3.33}$$

For simplicity, suppose

$$|v(F_3)||F_2^{x_{21}}||E_1^{x_{10}}| \gtrsim (|E_1||E_2||E_3|)^{1/3}. \tag{3.3.34}$$

Now we consider 3-by-3 matrices

$$\overline{A}_1 := \begin{pmatrix} (x_{10})_1 & (x_{10})_2 & (x_{10})_3 \end{pmatrix} \in E_1$$

with

$$\begin{pmatrix} (x_{10})_1 & (x_{10})_2 \end{pmatrix} = x_{10} \in \mathfrak{M}^{3\times 2}, \quad (x_{10})_3 \in E_1^{x_{10}};$$

and

$$\overline{A}_2 := \begin{pmatrix} (x_{21})_1 & (x_{21})_2 & (x_{21})_3 \end{pmatrix} \in E_2$$

with the condition

$$(x_{21})_1 = x_{21} \in \mathfrak{M}^{3\times 1}, \quad (x_{21})_2 \in F_2^{x_{21}}.$$

For any $\overline{A}_3 := \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in E_3$, for any constructed $\overline{A}_1, \overline{A}_2$ above,

$$\begin{aligned} s &\geq |\det(\overline{A}_1 + \overline{A}_2 + \overline{A}_3)| \\ &= |\det \begin{pmatrix} x_1 + (x_{10})_1 + (x_{21})_1 & x_2 + (x_{10})_2 + (x_{21})_2 & x_3 + (x_{10})_3 + (x_{21})_3 \end{pmatrix}|. \end{aligned}$$

So fix all columns except the 3rd column, we have

$$s \geq \sup_{(x_{10})_3 \in E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 + (x_{21})_1 & x_2 + (x_{10})_2 + (x_{21})_2 & x_3 + (x_{10})_3 + (x_{21})_3 \end{pmatrix}|.$$

Obviously,

$$s \geq \sup_{(x_{10})_3 \in \mathrm{co}E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 + (x_{21})_1 & x_2 + (x_{10})_2 + (x_{21})_2 & x_3 + (x_{10})_3 + (x_{21})_3 \end{pmatrix}|.$$

As before, by (3.3.22) we assume $\mathrm{co}E_1^{x_{10}}$ is an ellipsoid in $\mathbb{R}^3$, then there exists $T_0 \in O(3)$ such that $T_0\mathrm{co}E_1^{x_{10}}$ is an ellipsoid with principal axes parallel to the coordinate axes in $\mathbb{R}^3$. From (3.3.23) there eixsts an axis-parallel rectangle $L_{T_0\mathrm{co}E_1^{x_{10}}} \subset T_0\mathrm{co}E_1^{x_{10}}$ such that

$$|L_{T_0\mathrm{co}E_1^{x_{10}}}| \geq C|T_0\mathrm{co}E_1^{x_{10}}|.$$

Similar to (3.3.28), apply the invariance under $O(3)$ discussed in (3.3.19).

Thus

$$s \geq \sup_{(x_{10})_3 \in \mathrm{co}E_1^{x10}} |\det \begin{pmatrix} x_1 + (x_{10})_1 + (x_{21})_1 & x_2 + (x_{10})_2 + (x_{21})_2 & x_3 + (x_{10})_3 + (x_{21})_3 \end{pmatrix}|$$
$$= \sup_{(x_{10})_3 \in \mathrm{co}E_1^{x10}} |\det(M)|,$$

where

$$M = \begin{pmatrix} T_0(x_1 + (x_{10})_1 + (x_{21})_1) & T_0(x_2 + (x_{10})_2 + (x_{21})_2) & T_0(x_3 + (x_{10})_3 + (x_{21})_3) \end{pmatrix}.$$

Since $L_{T_0\mathrm{co}E_1^{x10}}$ is an axis-parallel rectangle in $\mathbb{R}^3$, it can be written as $A_1 \times A_2 \times A_3$, where A_1, A_2, A_3 are intervals in $\mathbb{R}$. Similar to the proof of Lemma 3.3.1 together with

$$\mathcal{S}(L_{T_0\mathrm{co}E_1^{x10}}) = \mathcal{S}(L_{T_0\mathrm{co}E_1^{x10}} + h) = A_1^* \times A_2^* \times A_3^*, \ \forall\ h \in \mathbb{R}^3,$$

applying Lemma 3.2.4 gives for any $\begin{pmatrix} x_1 & x_2 \end{pmatrix} \in v(E_3)$,

$$s \geq \sup_{(x_{10})_3 \in \mathcal{S}(L_{T_0\mathrm{co}E_1^{x10}})} |\det \begin{pmatrix} T_0(x_1 + (x_{10})_1 + (x_{21})_1) & T_0(x_2 + (x_{10})_2 + (x_{21})_2) & (x_{10})_3 \end{pmatrix}|.$$

Then fix all columns except the 2nd column,

$$s \geq \sup_{(x_{21})_2 \in F_2^{x21}} |\det \begin{pmatrix} T_0(x_1 + (x_{10})_1 + (x_{21})_1) & T_0(x_2 + (x_{10})_2 + (x_{21})_2) & (x_{10})_3 \end{pmatrix}|$$

holds for any $(x_{10})_3 \in \mathcal{S}(L_{T_0\mathrm{co}E_1^{x10}})$. Similarly, by the convex property of $|\det|$ function when fixing other columns

$$s \geq \sup_{(x_{21})_2 \in \mathrm{co}F_2^{x21}} |\det \begin{pmatrix} T_0(x_1 + (x_{10})_1 + (x_{21})_1) & T_0(x_2 + (x_{10})_2 + (x_{21})_2) & (x_{10})_3 \end{pmatrix}|,$$

By (3.3.22) we may assume $T_0\mathrm{co}F_2^{x21}$ is an ellipsoid in $\mathbb{R}^3$. Choose a rotation $T_1 \in O(3)$ such that $T_1T_0\mathrm{co}F_2^{x21}$ is an ellipsoid with principal axes parallel to the coordinate axes. From (3.3.23) there exists an axis-parallel rectangle $L_{T_1T_0\mathrm{co}F_2^{x21}} \subset T_1T_0\mathrm{co}F_2^{x21}$ such that

$$|L_{T_1T_0\mathrm{co}F_2^{x21}}| \geq C|T_1T_0\mathrm{co}F_2^{x21}|.$$

By the invariance of $O(3)$,

$$s \geq \sup_{(x_{21})_2 \in \mathrm{co}F_2^{x21}} |\det \begin{pmatrix} T_0(x_1 + (x_{10})_1 + (x_{21})_1) & T_0(x_2 + (x_{10})_2 + (x_{21})_2) & (x_{10})_3 \end{pmatrix}|$$
$$= \sup_{(x_{21})_2 \in \mathrm{co}F_2^{x21}} |\det \begin{pmatrix} T_1T_0(x_1 + (x_{10})_1 + (x_{21})_1) & T_1T_0(x_2 + (x_{10})_2 + (x_{21})_2) & T_1(x_{10})_3 \end{pmatrix}|.$$

Since $L_{T_1T_0\mathrm{co}F_2^{x21}}$ is an axis-parallel rectangle, together with

$$\mathcal{S}(L_{T_1T_0\mathrm{co}F_2^{x21}}) = \mathcal{S}(L_{T_1T_0\mathrm{co}F_2^{x21}} + h), \quad \forall\ h \in \mathbb{R}^3$$

apply Lemma 3.2.4 again to obtain

$$s \geq \sup_{\substack{(x_{10})_3 \in \mathcal{S}(L_{T_0 \mathrm{co} E_1^{x_{10}}}) \\ (x_{21})_2 \in \mathcal{S}(L_{T_1 T_0 \mathrm{co} F_2^{x_{21}}})}} |\det \begin{pmatrix} T_1T_0(x_1 + (x_{10})_1 + (x_{21})_1) & (x_{21})_2 & T_1(x_{10})_3 \end{pmatrix}|$$

holds for any $x_1 \in v(F_3) \subset \mathfrak{M}^{3\times 1}$. Lastly, applying (2.2.20) we conclude

$$\begin{aligned} s &\geq C|T_1T_0v(F_3) + T_1T_0(x_{10})_1 + T_1T_0(x_{21})_1|^{1/3}|\mathcal{S}(L_{T_1T_0\mathrm{co}F_2^{x_{21}}})|^{1/3}|T_1\mathcal{S}(L_{T_0\mathrm{co}E_1^{x_{10}}})|^{1/3} \\ &\geq C|v(F_3)|^{1/3}|\mathrm{co}F_2^{x_{21}})|^{1/3}|\mathrm{co}E_1^{x_{10}})|^{1/3}. \end{aligned}$$

This together with (3.3.34) implies

$$s \geq C|v(F_3)|^{1/3}|\mathrm{co}F_2^{x_{21}}|^{1/3}|\mathrm{co}E_1^{x_{10}}|^{1/3} \geq C|v(F_3)|^{1/3}|F_2^{x_{21}}|^{1/3}|E_1^{x_{10}}|^{1/3} \geq C(|E_1||E_2||E_3|)^{1/9}.$$

This completes (3.3.18) for $n = 3$.

For the general n, for each E_j, denote $F_{j0} = E_j$, $1 \leq j \leq n$. Given $1 \leq k \leq n-2$, let

$$F_{jk} = v(F_{j(k-1)}) \subset \mathfrak{M}^{n\times(n-k)},$$

then by (3.3.21) there exists fixed $x_{jk} \in v(F_{jk}) \subset \mathfrak{M}^{n\times(n-k-1)}$, $0 \leq k \leq n-2$, such that

$$|v(F_{jk})||F_{jk}^{x_{jk}}| \gtrsim |F_{jk}| = |v(F_{j(k-1)})|. \tag{3.3.35}$$

That is, for each E_j, $1 \leq j \leq n$, there exist $\{x_{j0}, \ldots, x_{j(n-2)}\}$ such that for each $k = 0, \ldots, n-2$

$$x_{jk} \in v(F_{jk}) \subset \mathfrak{M}^{n\times(n-k-1)},$$

and

$$|v(F_{j(n-2)})||F_{j(n-2)}^{x_{j(n-2)}}||F_{j(n-3)}^{x_{j(n-3)}}|\ldots|F_{j1}^{x_{j1}}||F_{j0}^{x_{j0}}| \gtrsim_n |E_j|. \tag{3.3.36}$$

It is not hard to see there exist $\{i_j\}_{j=1}^n$ with $1 \leq i_j \leq n$ and $i_j \neq i_k$ for $j \neq k$ such that

$$\begin{aligned} &(|v(F_{i_n(n-2)})||F_{i_{n-1}(n-2)}^{x_{i_{n-1}(n-2)}}||F_{i_{n-2}(n-3)}^{x_{i_{n-2}(n-3)}}|\ldots|F_{i_2 1}^{x_{i_2 1}}||F_{i_1 0}^{x_{i_1 0}}|)^n \\ \geq &\prod_{j=1}^n(|v(F_{j(n-2)})||F_{j(n-2)}^{x_{j(n-2)}}||F_{j(n-3)}^{x_{j(n-3)}}|\ldots|F_{j1}^{x_{j1}}||F_{j0}^{x_{j0}}|) \\ \gtrsim_n &\prod_{j=1}^n|E_j|. \end{aligned}$$

For simplicity, denote $i_j = j, 1 \leq j \leq n$. That is,

$$|v(F_{n(n-2)})||F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}||F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}|\ldots|F_{21}^{x_{21}}||F_{10}^{x_{10}}| \gtrsim_n \prod_{j=1}^n|E_j|^{1/n}. \tag{3.3.37}$$

To study the suprema, we consider the following n-by-n matrices

$$\overline{A}_1 := \begin{pmatrix} (x_{10})_1 & \ldots & (x_{10})_n \end{pmatrix} \in E_1$$

with $\begin{pmatrix} (x_{10})_1 & \dots & (x_{10})_{(n-1)} \end{pmatrix} = x_{10} \in \mathfrak{M}^{n\times(n-1)}$ and $(x_{10})_n \in F_{10}^{x_{10}}$;

$$\overline{A}_2 := \begin{pmatrix} (x_{21})_1 & \dots & (x_{21})_n \end{pmatrix} \in E_2$$

with $\begin{pmatrix} (x_{21})_1 & \dots & (x_{21})_{(n-2)} \end{pmatrix} = x_{21} \in \mathfrak{M}^{n\times(n-2)}$ and $(x_{21})_{n-1} \in F_{21}^{x_{21}}$. That is, construct $\{\overline{A}_1, \dots, \overline{A}_{n-1}\}$ such that for each $1 \le k \le n-1$

$$\overline{A}_k := \begin{pmatrix} (x_{k(k-1)})_1 & \dots & (x_{k(k-1)})_n \end{pmatrix} \in E_k,$$

with the condition that

$$\begin{pmatrix} x_{k(k-1)})_1 & \dots & (x_{k(k-1)})_{n-k} \end{pmatrix} = x_{k(k-1)} \in \mathfrak{M}^{n\times(n-k)}, \quad (x_{k(k-1)})_{n-k+1} \in F_{k(k-1)}^{x_{k(k-1)}}.$$

For any $\overline{A}_n := \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \in E_n$, for any constructed $\overline{A}_1, \dots, \overline{A}_{n-1}$ above,

$$\begin{aligned} s &\ge |\det(\overline{A}_1 + \dots + \overline{A}_{n-1} + \overline{A}_n)| \\ &= |\det \begin{pmatrix} x_1 + \sum_{k=1}^{n-1} (x_{k(k-1)})_1 & \dots & x_n + \sum_{k=1}^{n-1} (x_{k(k-1)})_n \end{pmatrix} |. \end{aligned}$$

Under the same notation as above, take the same arguments as in the case $n = 3$, then there exist $T_0, T_1 \in O(n)$

$$s \ge \sup_{\substack{(x_{10})_n \in \mathcal{S}(L_{T_0 \mathrm{co} F_{10}^{x_{10}}}) \\ (x_{21})_{(n-1)} \in \mathcal{S}(L_{T_1 T_0 \mathrm{co} F_{21}^{x_{21}}})}} |\det \begin{pmatrix} B & B' \end{pmatrix} |, \tag{3.3.38}$$

where

$$B = T_1 T_0 \begin{pmatrix} x_1 + \sum_{k=1}^{n-1} (x_{k(k-1)})_1 & \dots & x_{n-2} + \sum_{k=1}^{n-1} (x_{k(k-1)})_{n-2} \end{pmatrix} \in \mathfrak{M}^{n\times(n-2)},$$

and

$$B' = \begin{pmatrix} (x_{21})_{(n-1)} & T_1 (x_{10})_n \end{pmatrix} \in \mathfrak{M}^{n\times 2}.$$

Applying the same arguments again to (3.3.38), there exist $T_2 \in O(n)$

$$s \ge \sup_{\substack{(x_{10})_n \in \mathcal{S}(L_{T_0 \mathrm{co} F_{10}^{x_{10}}}) \\ (x_{21})_{(n-1)} \in \mathcal{S}(L_{T_1 T_0 \mathrm{co} F_{21}^{x_{21}}}) \\ (x_{32})_{n-2} \in \mathcal{S}(L_{T_2 T_1 T_0 \mathrm{co} F_{32}^{x_{32}}})}} |\det \begin{pmatrix} C & C' \end{pmatrix} |, \tag{3.3.39}$$

where

$$C = T_2 T_1 T_0 \begin{pmatrix} x_1 + \sum_{k=1}^{n-1} (x_{k(k-1)})_1 & \dots & x_{n-3} + \sum_{k=1}^{n-1} (x_{k(k-1)})_{n-3} \end{pmatrix} \in \mathfrak{M}^{n\times(n-3)},$$

and

$$C' = \begin{pmatrix} (x_{32})_{(n-2)} & T_2 (x_{21})_{(n-1)} & T_2 T_1 (x_{10})_n \end{pmatrix} \in \mathfrak{M}^{n\times 3}.$$

Keep repeating the same arguments above and finally we have there exist $T_0, \dots, T_{n-2} \in$

$O(n)$, such that for any $x_1 \in v(F_{n(n-2)}) \subset \mathfrak{M}^1$,

$$s \geq \sup_{\substack{(x_{10})_n \in \mathcal{S}(L_{T_0 \mathrm{co} F_{10}^{x_{10}}}) \\ (x_{21})_{(n-1)} \in \mathcal{S}(L_{T_1 T_0 \mathrm{co} F_{21}^{x_{21}}}) \\ \cdots\cdots \\ (x_{(n-1)(n-2)})_2 \in \mathcal{S}(L_{T_{n-2} T_{n-3} \dots T_0 \mathrm{co} F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}})}} |\det\begin{pmatrix} D & D' \end{pmatrix}|, \tag{3.3.40}$$

where $D \in \mathfrak{M}^{n\times 1}$, $D' \in \mathfrak{M}^{n\times(n-1)}$:

$$D = (T_{n-2} \dots T_0)(x_1 + \sum_{k=1}^{n-1} (x_{k(k-1)})_1),$$

$$D' = \begin{pmatrix} (x_{(n-1)(n-2)})_2 & T_{n-2}(x_{(n-2)(n-3)})_3 & (T_{n-2}T_{n-3})(x_{(n-3)(n-4)})_4 & \dots & (T_{n-2}\dots T_1)(x_{10})_n \end{pmatrix}.$$

Because of the invariance under $O(n)$ and the property of (3.3.23), applying (2.2.20) gives

$$s \geq C|v(F_{n(n-2)})|^{1/n} |\mathrm{co} F_{(n-1)(n-2)}^{x_{n-1)(n-2)}}|^{1/n} |\mathrm{co} F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}|^{1/n} \dots |\mathrm{co} F_{21}^{x_{21}}|^{1/n} |\mathrm{co} F_{10}^{x_{10}}|^{1/n}.$$

Obviously,

$$|\mathrm{co} F_{k(k-1)}^{x_{k(k-1)}}| \geq |F_{k(k-1)}^{x_{k(k-1)}}|,\ 1 \leq k \leq n-1.$$

This together with (3.3.37) implies

$$\begin{aligned} s &\geq C(|v(F_{n(n-2)})||\mathrm{co} F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}||\mathrm{co} F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}| \dots |\mathrm{co} F_{21}^{x_{21}}||F_{10}^{x_{10}}|)^{1/n} \\ &\geq C(|v(F_{n(n-2)})||F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}||F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}| \dots |F_{21}^{x_{21}}||F_{10}^{x_{10}}|)^{1/n} \\ &\geq C \prod_{j=1}^{n} |E_j|^{\frac{1}{n^2}}. \end{aligned}$$

This completes Theorem 3.3.5.

□

Corollary 3.3.6. *There exists a finite constant $\mathcal{A}_n, \mathcal{B}_n$ such that for any measurable set $E \subset \mathfrak{M}^{n\times n}$ of finite measure, for any non-zero scalar $\lambda_j \in \mathbb{R}$, $j = 1, \dots, n$,*

$$(\prod_{j=1}^{n} |\lambda_j|)|E|^{\frac{1}{n}} \leq \mathcal{A}_n \sup_{\substack{A_j \in E \\ j=1,\dots,n}} |\det(\lambda_1 A_1 + \dots + \lambda_n A_n)|. \tag{3.3.41}$$

If E is a compact convex set in $\mathfrak{M}^{n\times n}$, then

$$|E|^{\frac{1}{n}} \leq \mathcal{B}_n \sup_{A\in E} |\det(A)|. \tag{3.3.42}$$

Proof. To see (3.3.41), let $E_j = \lambda_j E$. Applying Theorem 3.3.5 gives

$$\prod_{j=1}^{n} |\lambda_j E|^{\frac{1}{n^2}} \leq \mathcal{C}_n \sup_{\substack{A_j \in E \\ j=1,\dots,n}} |\det(\lambda_1 A_1 + \dots + \lambda_n A_n)|,$$

which implies (3.3.41). In particular, if $E \subset \mathfrak{M}^{n\times n}$ is a compact convex set, setting $\lambda_j = \frac{1}{n}$, $j = 1, \ldots, n$, it follows from (3.3.41) that

$$(\frac{1}{n})^n |E|^{1/n} \leq \mathcal{A}_n \sup_{\substack{A_j \in E \\ j=1,\ldots,n}} |\det(\frac{1}{n}A_1 + \cdots + \frac{1}{n}A_n)|.$$

On the other hand, since E is convex,

$$\sup_{A\in E} |\det(A)| \geq \sup_{\substack{A_j \in E \\ j=1,\ldots,n}} |\det(\frac{1}{n}A_1 + \cdots + \frac{1}{n}A_n)|.$$

Thus we get (3.3.42).

□

Here we give a direct way to see Lemma 13.2 [15] which follows from (3.3.42). Let $E \subset \mathfrak{M}^{n\times n}$ be a measurable set. The inequality (1.3.16) in Lemma 13.2 has translation invariance property, so we assume that $0 \in E$. Given any matrices $A_1, \ldots, A_{n^2}$ in E, from (3.3.42) it follows that

$$|\mathrm{co}\{0, A_1, \ldots, A_{n^2}\}|^{\frac{1}{n}} \lesssim_n \sup_{A \in \mathrm{co}\{0, A_1, \ldots, A_{n^2}\}} |\det(A)|, \tag{3.3.43}$$

By (2.2.20), there exist $A_1, \ldots, A_{n^2}$ such that

$$|E| \lesssim_n |\mathrm{co}\{0, A_1, \ldots, A_{n^2}\}|,$$

together with (3.3.43) we obtain that

$$|E|^{\frac{1}{n}} \lesssim_n \sup_{A \in \mathrm{co}\{0, A_1, \ldots, A_{n^2}\}} |\det(A)|. \tag{3.3.44}$$

For any convex set $F \subset \mathfrak{M}^{n\times n}$

$$\sup_{A \in \mathrm{co}\{0, F\}} |\det(A)| = \sup_{A\in F} |\det(A)|,$$

since $|\det(\lambda A)| = \lambda^n |\det(A)| \leq |\det(A)|$ for any $\lambda \in [0, 1]$. So

$$\sup_{A \in \mathrm{co}\{0, A_1, \ldots, A_{n^2}\}} |\det(A)| = \sup_{A \in \mathrm{co}\{A_1, \ldots, A_{n^2}\}} |\det(A)|. \tag{3.3.45}$$

Denote $A^{(k)}$ by the k-th column vector of the matrix A, $1 \leq k \leq n$. Then there exist $\widetilde{A}_1, \ldots, \widetilde{A}_n \in \{A_1, \ldots, A_{n^2}\}$ ($\widetilde{A}_i$, $\widetilde{A}_j$ might be the same matrix), such that for any $\{\lambda_1, \ldots, \lambda_{n^2}\}$ satisfying $\sum\limits_{j=1}^{n^2} \lambda_j = 1$ and $0 \leq \lambda_j \leq 1$,

$$|\det(\lambda_1 A_1 + \cdots + \lambda_{n^2} A_{n^2})| \leq |\sum_{\substack{i_j \in \{1,\ldots,n\} \\ i_j \neq i_k, \forall j \neq k}} \det(\widetilde{A}_{i_1}^{(1)}, \ldots, \widetilde{A}_{i_n}^{(n)})| \tag{3.3.46}$$

holds, this is because

$$\sum_{1\le l_1,\dots,l_n\le n^2}\lambda_{l_1}\dots\lambda_{l_n}\le\sum_{1\le l_1,\dots,l_{n-1}\le n^2}\lambda_{l_1}\dots\lambda_{l_{n-1}}\le\dots\le\sum_{1\le l_1,l_2\le n^2}\lambda_{l_1}\lambda_{l_2}\le\sum_{1\le l_1\le n^2}\lambda_{l_1}=1.$$

Hence from (3.3.44)-(3.3.46)

$$|E|^{\frac{1}{n}}\lesssim_n|\sum_{\substack{i_j\in\{1,\dots,n\}\\ i_j\ne i_k,\forall j\ne k}}\det(\widetilde{A}^{(1)}_{i_1},\dots,\widetilde{A}^{(n)}_{i_n})|. \tag{3.3.47}$$

As mentioned in the proof of Lemma 13.2 [15], $\sum\limits_{\substack{i_j\in\{1,\dots,n\}\\ i_j\ne i_k,\forall j\ne k}}\det(\widetilde{A}^{(1)}_{i_1},\dots,\widetilde{A}^{(n)}_{i_n})$ is $\mathbb{Z}$-linear combination of $\{\det(\sum\limits_{j=1}^{n}s_j\widetilde{A}_j):s_j\in\{0,1\}\}$. This gives (1.3.19):

$$|E|^{\frac{1}{n}}\lesssim_n\sup_{\substack{A_1,\dots,A_n\in E\\ s_1,\dots,s_n\in\{0,1\}}}|\det(s_1A_1+\dots+s_nA_n)|.$$

Obviously, (3.3.42) is not affine invariant. The following example shows balls or ellipsoids are not the optimisers.

Example 3.3.7. (i) Let $n=2$, $E=B(0,r)$, $A=\begin{pmatrix}a&c\\b&d\end{pmatrix}\in E$. Then $\sup\limits_{A\in E}|\det(A)|=\frac{r^2}{2}$ by calculation. Consider the ellipsoid F in $\mathbb{R}^4$ with $|F|=|B(0,r)|$,

$$F=\{\begin{pmatrix}a&c\\b&d\end{pmatrix}:\frac{a^2}{l_1^2}+\frac{b^2}{l_2^2}+\frac{c^2}{l_3^2}+\frac{d^2}{l_4^2}\le1\}.$$

It is easy to obtain $\sup\limits_{A\in F}|\det(A)|\ge\frac{l_1l_4+l_2l_3}{4}\ge\frac{r^2}{2}$ by GM-AM inequality.

(ii) Let $r=1$. Since $A\mapsto|\det(A)|$ is a continuous function on $E=B(0,1)$ under the natural topology on Euclidean space $\mathbb{R}^4$, there exists $0<\delta<\frac{1}{25}$ such that $|\det(A)|\le\frac14$ for all $A\in E$ satisfying

$$|A-\begin{pmatrix}1&0\\0&0\end{pmatrix}|=(a-1)^2+b^2+c^2+d^2\le2\delta.$$

Then for all $A\in E$ satisfying $\sqrt{1-\delta}\le a\le1$, we have

$$b^2+c^2+d^2\le1-a^2\le1-(1-\delta)=\delta.$$

Thus

$$|A-\begin{pmatrix}1&0\\0&0\end{pmatrix}|=(a-1)^2+b^2+c^2+d^2\le(1-\sqrt{1-\delta})^2+\delta\le2\delta$$

which implies that $|\det(A)|\le\frac14$ for any $A\in E$ satisfying $\sqrt{1-\delta}\le a\le1$.

Let $P = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ with $p = \frac{1}{\sqrt{1-\delta}}$ and then consider $\sup\limits_{A \in \mathrm{co}\{P \cup E\}} |\det(A)|$,

$$\begin{aligned}
\sup_{A \in \mathrm{co}\{P \cup E\}} |\det(A)| &= \sup_{A \in E, \lambda \in [0,1]} |\det(\lambda A + (1-\lambda)P)| \\
&= \sup_{A \in E, \lambda \in [0,1]} |\det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d + (1-\lambda)p \end{pmatrix}| \\
&= \sup_{A \in E, \lambda \in [0,1]} |\det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{pmatrix} + \det \begin{pmatrix} \lambda a & 0 \\ \lambda b & (1-\lambda)p \end{pmatrix}|.
\end{aligned}$$

When $a \notin [\sqrt{1-\delta}, 1]$,

$$\begin{aligned}
&\sup_{A \in E, \lambda \in [0,1]} |\det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{pmatrix} + \det \begin{pmatrix} \lambda a & 0 \\ \lambda b & (1-\lambda)p \end{pmatrix}| \\
&\leq \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{2} + \lambda(1-\lambda)ap \\
&\leq \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{2} + \lambda(1-\lambda)\sqrt{1-\delta}\frac{1}{\sqrt{1-\delta}} \\
&= \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{2} + \lambda(1-\lambda) \leq \frac{1}{2}.
\end{aligned}$$

When $a \in [\sqrt{1-\delta}, 1]$,

$$\begin{aligned}
&\sup_{A \in E, \lambda \in [0,1]} |\det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{pmatrix} + \det \begin{pmatrix} \lambda a & 0 \\ \lambda b & (1-\lambda)p \end{pmatrix}| \\
&\leq \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{4} + \lambda(1-\lambda)p \\
&= \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{4} + \lambda(1-\lambda)\frac{1}{\sqrt{1-\delta}}.
\end{aligned}$$

It is easy to see for $0 < \delta < \frac{1}{25}$ given above,

$$\sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{4} + \lambda(1-\lambda)\frac{1}{\sqrt{1-\delta}} \leq \frac{1}{2}.$$

Therefore,

$$\sup_{A \in \mathrm{co}\{P \cup E\}} |\det(A)| = \sup_{A \in E} |\det(A)|,$$

which implies balls can not be the optimisers.

Remark 3.3.8. Let $E \subset \mathfrak{M}^{n \times n}$ be a compact convex set. If we compare the maximal volume of simiplicies $\sup\limits_{A_0, \ldots, A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \ldots, A_{n^2}\})$ contained in E with the $\sup\limits_{A \in E} |\det(A)|$, it follows from (3.3.42) that

$$\sup_{A_0, \ldots, A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \ldots, A_{n^2}\}) \lesssim_n \sup_{A \in E} |\det(A)|^n. \tag{3.3.48}$$

Indeed by John ellipsoids, it is enough to consider the case when E is a ellipsoid in $\mathfrak{M}^{n\times n}$. For any ellpsoid

$$E \equiv \{x \in \mathbb{R}^{n^2} : \sum_{i}^{n^2} \frac{|\langle x - x_0, \omega_i \rangle|^2}{l_i^2} \leq 1\},$$

where $x_0 \in \mathbb{R}^{n^2}$, $\{\omega_i\}$ is an orthonormal basis in $\mathbb{R}^{n^2}$. By the affine invariance of $\sup\limits_{A_0,\dots,A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \dots, A_{n^2}\})$, it is enough to see balls centred at 0. Apply the Hadamard inequality, for any $A_j \in B(0,r) \subset \mathbb{R}^{n^2}$, $j = 0, \dots, n^2$

$$\mathrm{vol}(\mathrm{co}\{A_0, \dots, A_{n^2}\}) \leq |A_0 - A_1||A_0 - A_2| \dots |A_0 - A_{n^2}| \lesssim_n r^{n^2} \sim |B(0,r)|.$$

Hence for any ellipsoid $E \subset \mathbb{R}^{n^2}$,

$$\sup_{A_0,\dots,A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \dots, A_{n^2}\}) \lesssim_n |E|.$$

On the other hand, by (3.3.42)

$$|E| \lesssim_n \sup_{A \in E} |\det(A)|^n.$$

Therefore, we have the following relation

$$\sup_{A_0,\dots,A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \dots, A_{n^2}\}) \lesssim_n \sup_{A \in E} |\det(A)|^n.$$

Similarly, we have

$$\sup_{A_1,\dots,A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{0, A_1, \dots, A_{n^2}\}) \lesssim_n \sup_{A \in E} |\det(A)|^n.$$

If $0 \in E$, it is true which mainly due to the Hadamard inequality and the $\mathrm{GL}_n(\mathbb{R})$ invariance of $\sup\limits_{A_1,\dots,A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{0, A_1, \dots, A_{n^2}\})$. If $0 \notin E$, the relation above still holds because of the fact

$$\sup_{A \in E} |\det(A)|^n = \sup_{A \in \mathrm{co}\{0,E\}} |\det(A)|^n.$$

Chapter 4

Sharp Versions of Geometric Inequalities

In this chapter we make use of competing symmetries and rearrangement inequalities Theorem 3.2.3, Theorem 3.2.5 and Theorem 3.3.4, in order to find the optimal constants for the following geometric inequalities. Competing symmetries has been applied before to obtain the sharp Hardy-Littlewood -Sobolev inequality and the sharp Sobolev inequality (see [8], [9], [11], and [32]). Similarly, in the following two sections we use symmetries together with rearrangement inequalities studied in Chapter 3 to construct a strongly convergent sequence of functions in L^p.

4.1 Sharp Constants for Bilinear Geometric Inequalities

Theorem 4.1.1. *Let $0 < p < \infty$ and f, g be nonnegative measurable functions in $L^p(\mathbb{R}^n)$. For the geometric inequality*

$$\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \sup_{x,y} f(x)g(y)|x-y|^{\frac{2n}{p}}, \tag{4.1.1}$$

the minimum constant $C_{p,n}$ is obtained for $f = const \cdot h$, and $g = const \cdot h$, where

$$h(x) = (1+|x|^2)^{-\frac{n}{p}}.$$

Later we can see the sharp constant $C_{p,n} = 2^{-\frac{2n}{p}}|\mathbb{S}^n|^{\frac{2}{p}}$, where $|\mathbb{S}^n|$ is the surface area of the unit sphere $\mathbb{S}^n$.

Let $q \in (0, \infty)$, $p \in (1, \infty)$. Form

$$\sup_{x,y} f(x)^{\frac{p}{q}} g(y)^{\frac{p}{q}} |x-y|^{\frac{2n}{q}} = (\sup_{x,y} f(x)g(y)|x-y|^{\frac{2n}{p}})^{\frac{p}{q}},$$

and

$$\|f^{\frac{p}{q}}\|_{L^q(\mathbb{R}^n)} = (\|f\|_{L^p(\mathbb{R}^n)})^{\frac{p}{q}},\ \ \|g^{\frac{p}{q}}\|_{L^q(\mathbb{R}^n)} = (\|g\|_{L^p(\mathbb{R}^n)})^{\frac{p}{q}},$$

we observe that if f, g is a pair of extremals for $p \in (1, \infty)$, then $f^{\frac{p}{q}}$, $g^{\frac{p}{q}}$ is a pair of extremals for any $q \in (0, \infty)$. So it suffices to study the extremals for the case when $1 < p < \infty$.

From Theorem 3.2.3, it suffices to seek optimisers amongst the class of all symmetric decreasing functions.

Let $\mathcal{S}$ be the stereographic projection from $\mathbb{R}^n$ to the unit sphere $\mathbb{S}^n$ with

$$\mathcal{S}(x) = (\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}),$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. So

$$\mathcal{S}^{-1}(s) = (\frac{s_1}{1+s_{n+1}}, \ldots, \frac{s_n}{1+s_{n+1}}).$$

For $f \in L^p(\mathbb{R}^n)$, define

$$(\mathcal{S}^* f)(s) := |J_{\mathcal{S}^{-1}}(s)|^{1/p} f(\mathcal{S}^{-1}(s)), \ (\mathcal{S}^* g)(t) := |J_{\mathcal{S}^{-1}}(t)|^{1/p} g(\mathcal{S}^{-1}(t)), \tag{4.1.2}$$

where $J_{\mathcal{S}^{-1}}$ is the Jacobian determinant of the map $\mathcal{S}^{-1}$

$$|J_{\mathcal{S}^{-1}}(s)| = (\frac{1}{1+s_{n+1}})^n = \frac{1}{2^n}(1 + |\mathcal{S}^{-1}(s)|^2)^n. \tag{4.1.3}$$

Then we have the invariance of the geometric inequality under the stereographic projection shown as the following lemma.

Lemma 4.1.2. *For nonnegative measurable functions f, $g \in L^p(\mathbb{R}^n)$, denote $F(s) = (\mathcal{S}^* f)(s), G(t) = (\mathcal{S}^* g)(t)$. Then*

$$\sup_{x,y \in \mathbb{R}^n} f(x)g(y)|x-y|^{\frac{2n}{p}} = \sup_{s,t \in \mathbb{S}^n} F(s)G(t)|s-t|^{\frac{2n}{p}},$$

and

$$\|F\|_{L^p(\mathbb{S}^n)} = \|f\|_{L^p(\mathbb{R}^n)}, \ \|G\|_{L^p(\mathbb{S}^n)} = \|g\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $x = \mathcal{S}^{-1}(s), y = \mathcal{S}^{-1}(t)$. It follows from Lieb-Loss [32] that

$$|J_{\mathcal{S}^{-1}}(s)| = (\frac{1+|\mathcal{S}^{-1}(s)|^2}{2})^n,$$

and

$$|x-y| = |s-t|(\frac{1+|x|^2}{2})^{1/2}(\frac{1+|y|^2}{2})^{1/2} = |J_{\mathcal{S}^{-1}}(s)|^{\frac{1}{2n}}|J_{\mathcal{S}^{-1}}(t)|^{\frac{1}{2n}}|s-t|.$$

Hence

$$\begin{aligned} \sup_{s,t \in \mathbb{S}^n} F(s)G(t)|s-t|^{\frac{2n}{p}} &= \sup_{s,t \in \mathbb{S}^n} |J_{\mathcal{S}^{-1}}(s)|^{1/p} f(\mathcal{S}^{-1}(s))|J_{\mathcal{S}^{-1}}(t)|^{1/p} g(\mathcal{S}^{-1}(t))|s-t|^{\frac{2n}{p}} \\ &= \sup_{x,y \in \mathbb{R}^n} f(x)g(y)|x-y|^{\frac{2n}{p}}. \end{aligned}$$

The invariance of L^p norm can be obtained as follows,

$$\|f\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p} = (\int_{\mathbb{S}^n} |f(\mathcal{S}^{-1}(s))|^p |J_{\mathcal{S}^{-1}}(s)| ds)^{1/p} = (\int_{\mathbb{S}^n} |F(s)|^p ds)^{1/p}.$$

Applying a similar argument implies $\|g\|_{L^p(\mathbb{R}^n)} = \|G\|_{L^p(\mathbb{S}^n)}$.

□

Now we are ready to prove the sharp case of inequality (4.1.1).

Proof of Theorem 4.1.1

For $f \in L^p(\mathbb{R}^n)$, consider a rotation $D : \mathbb{S}^n \to \mathbb{S}^n$ with

$$D(s) = (s_1, \ldots, s_{n-1}, s_{n+1}, -s_n).$$

Specifically, it is a rotation of the sphere by 90° which keeps the other basis vectors fixed except n-th and $(n+1)$-th vectors in the direction of mapping the $(n+1)$-th vector $e_{n+1} = (0, \ldots, 0, 1)$ to n-th vector $e_n = (0, \ldots, 0, 1, 0)$.
Define $(D^*F)(s) = |J_{D^{-1}}(s)|^{\frac{1}{p}} F(D^{-1}(s)) = F(D^{-1}(s))$ for any $F \in L^p(\mathbb{S}^n)$. Then

$$\|D^*F\|_p = \|F\|_p,$$

which shows D^* is norm preserving.

We consider the new function $(\mathcal{S}^*)^{-1} D^* \mathcal{S}^* f$, where $(\mathcal{S}^* f)(s)$ is the same as (4.7). Denote $(\mathcal{S}^* f)(s)$ by $F(s)$, and let $x = \mathcal{S}^{-1}(s)$. From the discussion above, we have already shown

$$F(s) = (\frac{1+|x|^2}{2})^{\frac{n}{p}} f(x).$$

The definition of D and $\mathcal{S}$ implies

$$D^{-1}(s) = (s_1, \ldots, s_{n-1}, -s_{n+1}, s_n) = (\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2}, \frac{2x_n}{1+|x|^2}).$$

Then

$$(D^* \mathcal{S}^* f)(s) = (D^* F)(s) = F(D^{-1}(s)) = (\frac{1+|x|^2}{|x+e_n|})^{\frac{n}{p}} f(\frac{2x_1}{|x+e_n|^2}, \ldots, \frac{2x_{n-1}}{|x+e_n|^2}, \frac{|x|^2-1}{|x+e_n|^2}),$$

this is because

$$\mathcal{S}^{-1}(D^{-1}(s)) = (\frac{2x_1}{|x+e_n|^2}, \ldots, \frac{2x_{n-1}}{|x+e_n|^2}, \frac{|x|^2-1}{|x+e_n|^2}).$$

Finally we find

$$\begin{aligned}
(\mathcal{S}^{*-1} D^* \mathcal{S}^* f)(x) &= (\frac{1+|x|^2}{2})^{-\frac{n}{p}} F(D^{-1}(s)) \\
&= (\frac{2}{|x+e_n|})^{\frac{n}{p}} f(\frac{2x_1}{|x+e_n|^2}, \ldots, \frac{2x_{n-1}}{|x+e_n|^2}, \frac{|x|^2-1}{|x+e_n|^2}).
\end{aligned}$$

Briefly speaking, we lift f to the sphere by (4.1.2) first, then rotate it by $90°$ in a specific direction which maps the north pole e_{n+1} to e_n, lastly push back to $\mathbb{R}^n$. For simplicity we denote $\mathcal{S}^{*-1}D^*\mathcal{S}^*f$ by $\mathcal{D}f$.

Let $f \in L^p(\mathbb{R}^n)$. Applying the transformation $\mathcal{D}$ and the symmetric rearrangement to f many times gives the sequence $\{f_k\}_{k\in\mathbb{N}}$. Specifically, $f_0 = f$, $f_k = (\mathcal{R}\mathcal{D})^k f$. Note that both $\mathcal{D}$ and $\mathcal{R}$ are norm-preserving. This is because Lemma 4.1.2 implies

$$\|\mathcal{S}^{*-1}D^*\mathcal{S}^*f\|_p = \|D^*\mathcal{S}^*f\|_p.$$

Due to the norm preserving property of D^*, we have

$$\|D^*\mathcal{S}^*f\|_p = \|\mathcal{S}^*f\|_p$$

So apply Lemma 4.1.2 again to get

$$\|\mathcal{S}^{*-1}D^*\mathcal{S}^*f\|_p = \|\mathcal{S}^*f\|_p = \|f\|_p.$$

It follows from Theorem 4.6 in Lieb-Loss [32] that for all $f \in L^p(\mathbb{R}^n)$, the sequence f_k converges to h_f in L^p norm as $k \to \infty$. Here

$$h_f = c\ h, \quad h(x) = (1 + |x|^2)^{-\frac{n}{p}}$$

and c is the constant such that $\|f\|_p = \|h_f\|_p$, so the constant c is

$$c = 2^{\frac{n}{p}}|\mathbb{S}^n|^{-1/p}\|f\|_p,$$

where $|\mathbb{S}^n|$ means the area of unit sphere in $\mathbb{R}^{n+1}$.

For the reader's convenience, in the following we will sketch it. First it suffices to prove f_k converges to h_f in L^p as $k \to \infty$ for a dense set of functions in $L^p(\mathbb{R}^n)$. So we consider the bounded functions that vanish outside a bounded set, since they are dense in $L^p(\mathbb{R}^n)$. Then there exists a constant C such that $f(x) \le Ch_f(x)$ for almost every $x \in \mathbb{R}^n$. Note that $\mathcal{R}$ and $\mathcal{D}$ are order-preserving, then we have $f_k(x) \le Ch_f(x)$ for almost every$x$, and every k. Applying Helly's selection principle, there exists a subsequence f_{k_l} such that $f_{k_l}(x)$ converges to a symmetric decreasing function $g(x)$ almost everywhere as $l \to \infty$. By the dominated convergence theorem, $g \in L^p$. We define $A := \inf_k \|h_f - f_k\|_p$. It follows from the nonexpansivity of rearrangement $\mathcal{R}$ stated in Proposition 3.1.1

$$\|\mathcal{R}f - \mathcal{R}g\|_p \le \|f - g\|_p, \tag{4.1.4}$$

$$\|\mathcal{D}f - \mathcal{D}g\|_p = \|f - g\|_p, \tag{4.1.5}$$

and the invariance of h_f that $\|h_f - f_k\|_p$ decreases monotonically. Thus

$$A = \inf_k \|h_f - f_k\|_p = \lim_{k\to\infty} \|h_f - f_k\|_p.$$

Applying the invariance of h_f, (4.1.4) and (4.1.5) once again yields

$$\begin{aligned} A &= \lim_{l} \|h_f - f_{k_l+1}\|_p = \|h_f - \mathcal{R}\mathcal{D}g\|_p = \|\mathcal{R}\mathcal{D}h_f - \mathcal{R}\mathcal{D}g\|_p \\ &\leq \|\mathcal{D}h_f - \mathcal{D}g\|_p = \|h_f - g\|_p = A, \end{aligned} \tag{4.1.6}$$

then we must have equality everywhere

$$\|h_f - \mathcal{R}\mathcal{D}g\|_p = \|h_f - g\|_p = A.$$

Since h_f is strictly symmetric decreasing, from the Proposition 3.1.1 we have

$$\mathcal{R}\mathcal{D}g = \mathcal{D}g.$$

This together with the fact that $\mathcal{R}g = g$ gives $g = ch$. If g is symmetric decreasing, by the definition of $\mathcal{S}^*$, $(\mathcal{S}^*g)(s)$ is a function of s_{n+1}. That it,

$$(\mathcal{S}^*g)(s) = \phi(s_{n+1}),$$

where ϕ is defined on $\mathbb{S}^n$. Because $\mathcal{D}g$ is also symmetric decreasing, there exists some function ψ defined on $\mathbb{S}^n$ such that

$$(\mathcal{S}^*g)(\mathcal{D}^{-1}s) = \psi(s_{n+1}).$$

Therefore,

$$\phi(s_{n+1}) = (\mathcal{S}^*g)(s) = (\mathcal{S}^*g)(\mathcal{D}^{-1}\mathcal{D}s) = \psi((\mathcal{D}s)_{n+1}) = \psi(-s_n),$$

which means $\mathcal{S}^*g$ is a constant on $\mathbb{S}^n$ and thus $g = ch$. It is easily seen that $c = \|f\|_p$, so $g = h_f$, and $A = 0$. Thus f_k converges strongly to h_f in $L^p(\mathbb{R}^n)$.

Since f_k converges to h_f in L^p norm for all f $\in L^p(\mathbb{R}^n)$, there exist subsequences $\{f_{k_l}\}, \{g_{k_l}\}$ such that $f_{k_l} \to h_f$ and $g_{k_l} \to h_g$ pointwise almost everywhere as $l \to \infty$. Clearly, Theorem 3.2.3, Lemma 4.1.2 and the rearrangement property

$$(f^p)^* = (f^*)^p, \quad \text{for} \quad 0 < p < \infty$$

indicate that $\sup\limits_{x,y} f_k(x)g_k(y)|x-y|^{\frac{2n}{p}}$ decreases monotonically as k grows. Hence for all x, y, k_l

$$f_{k_l}(x)g_{k_l}(y)|x-y|^{\frac{2n}{p}} \leq \sup_{x,y} f(x)g(y)|x-y|^{\frac{2n}{p}} < \infty.$$

Together with the dominated convergence theorem it follows that

$$f_{k_l}(x)g_{k_l}(y)|x-y|^{\frac{2n}{p}} \xrightarrow{weak^*} h_f(x)h_g(y)|x-y|^{\frac{2n}{p}}$$

in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ as $l \to \infty$.
Note that L^∞ norm is weak* lower semicontinuous, which follows from the property that in dual space X^* of a Banach space X,

$$\text{If } x_n^* \xrightarrow{weak^*} x^* \text{ as } n \to \infty, \text{ then } \|x^*\|_{X^*} \leq \liminf_{n\to\infty} \|x_n^*\|_{X^*}.$$

Hence we have

$$\sup_{x,y} h_f(x)h_g(y)|x-y|^{\frac{2n}{p}} \le \liminf_{l\to\infty} (\sup_{x,y} f_{k_l}(x)g_{k_l}(y)|x-y|^{\frac{2n}{p}}) = \inf_{l} (\sup_{x,y} f_{k_l}(x)g_{k_l}(y)|x-y|^{\frac{2n}{p}}).$$

Therefore for every $f, g \in L^p(\mathbb{R}^n)$ and every k_l

$$\frac{\sup_{x,y} f(x)g(y)|x-y|^{\frac{2n}{p}}}{\|f\|_p\|g\|_p} \ge \frac{\sup_{x,y} f_{k_l}(x)g_{k_l}(y)|x-y|^{\frac{2n}{p}}}{\|f_{k_l}\|_p\|g_{k_l}\|_p} \ge \frac{\sup_{x,y} h_f(x)h_g(y)|x-y|^{\frac{2n}{p}}}{\|h_f\|_p\|h_g\|_p},$$

because of the norm-preserving of $\mathcal{RD}$.
Obviously,

$$\frac{\sup_{x,y} h_f(x)h_g(y)|x-y|^{\frac{2n}{p}}}{\|h_f\|_p\|h_g\|_p} = \frac{\sup_{x,y} h(x)h(y)|x-y|^{\frac{2n}{p}}}{\|h\|_p^2}.$$

□

Therefore, the conformally invariant property of (4.1.1) implies that if f and g are the same conformal transformation of h, equality still holds. However, here we can not characterise the optimisers.

From the sharp version for $\mathbb{R}^n$ case in Theorem 4.1.1 together with (4.1.2), (4.1.3) and the conformally invariant property in Lemma 4.1.2, it follows that the geometric inequality (4.1.1) has conformally equivalent form on the unit sphere $\mathbb{S}^n$ as follows.

Theorem 4.1.3. *For $0 < p < \infty$, let F, G be nonnegative functions in $L^p(\mathbb{S}^n)$. Then*

$$\|F\|_{L^p(\mathbb{S}^n)}\ \|G\|_{L^p(\mathbb{S}^n)} \le B_{p,n} \sup_{s,t\in\mathbb{S}^n} F(s)G(t)\ |s-t|^{\frac{2n}{p}}. \tag{4.1.7}$$

The best constant $B_{p,n}$ is obtained for F, G are constant functions, and the corresponding $B_{p,n} = 2^{-\frac{2n}{p}}|\mathbb{S}^n|^{\frac{2}{p}}$.

Meanwhile, let $\mathbb{H}^n$ be the hyperbolic space in $\mathbb{R}^{n+1}$:

$$\mathbb{H}^n = \{q = (q_1, \ldots, q_n, q_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : q_1^2 + \cdots + q_n^2 - q_{n+1}^2 = -1)\},$$

with the Lorenz group $O(1,n)$ invariant measure $d\nu(q)$. We find the geometric inequality (4.1.1) also has the conformally equivalent form in $\mathbb{H}^n$ space as shown in the following theorem.

Theorem 4.1.4. *For $0 < p < \infty$, let F, G be nonnegative functions in $L^p(\mathbb{H}^n)$. Then*

$$\|F\|_{L^p(\mathbb{H}^n)}\ \|G\|_{L^p(\mathbb{H}^n)} \le E_{p,n} \sup_{q,t} F(q)G(t)\ |qt-1|^{\frac{n}{p}}, \tag{4.1.8}$$

$qt = -q_1t_1 - \cdots - q_nt_n + q_{n+1}t_{n+1}$. *The best constant* $E_{p,n}$ *is obtained when* $F = const \cdot H$, $G = const \cdot H$, *where*

$$H(q) = |q_{n+1}|^{-\frac{n}{p}},\ q = (q_1, \dots, q_n, q_{n+1}).$$

Proof. Consider the stereographic projection $\mathcal{H}$ which is conformal transformation from $\mathbb{R}^n \backslash \{|x| = 1\}$ to $\mathbb{H}^n$ as

$$\mathcal{H}(x) = (\frac{2x_1}{1-|x|^2}, \dots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2}),$$

so

$$\mathcal{H}^{-1}(q) = (\frac{q_1}{1+q_{n+1}}, \dots, \frac{q_n}{1+q_{n+1}}).$$

Let $x = \mathcal{H}^{-1}(q), y = \mathcal{H}^{-1}(t)$. From Lieb-Loss [32], the Jacobian determinant of the map $\mathcal{H}^{-1}$ is

$$|J_{\mathcal{H}^{-1}}(q)| = (\frac{1-|\mathcal{H}^{-1}(q)|^2}{2})^n,$$

and

$$|x-y| = (\frac{|1-|x|^2|}{2})^{1/2}(\frac{|1-|y|^2|}{2})^{1/2}|qt-1|^{1/2} = |J_{\mathcal{H}^{-1}}(q)|^{\frac{1}{2n}}|J_{\mathcal{H}^{-1}}(t)|^{\frac{1}{2n}}|qt-1|^{1/2},$$

where $qt = -q_1t_1 - \cdots - q_nt_n + q_{n+1}t_{n+1}$.
Define

$$F(q) := |J_{\mathcal{H}^{-1}}(q)|^{1/p}f(\mathcal{H}^{-1}(q)),\ G(t) := |J_{\mathcal{H}^{-1}}(t)|^{1/p}g(\mathcal{H}^{-1}(t)).$$

Thus from the above, we easily get the conformal invariance as follows.

$$\begin{aligned}\sup_{q,t} F(q)G(t)|qt-1|^{\frac{n}{p}} &= \sup_{q,t} |J_{\mathcal{H}^{-1}}(q)|^{1/p}f(\mathcal{H}^{-1}(q))|J_{\mathcal{H}^{-1}}(t)|^{1/p}g(\mathcal{H}^{-1}(t))|qt-1|^{\frac{n}{p}} \\ &= \sup_{x,y} f(x)g(y)|x-y|^{\frac{2n}{p}},\end{aligned}$$

and

$$\|f\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p} = (\int_{\mathbb{H}^n} |f(\mathcal{H}^{-1}(q))|^p |J_{\mathcal{H}^{-1}}(q)| dq)^{1/p} = (\int_{\mathbb{H}^n} |F(q)|^p dq)^{1/p}.$$

Applying a similar argument gives $\|g\|_{L^p(\mathbb{R}^n)} = \|G\|_{L^p(\mathbb{H}^n)}$.

When $f(x) = c(1+|x|^2)^{-\frac{n}{p}}$, $F(q) = c\ (\frac{1-|x|^2}{1+|x|^2})^{\frac{n}{p}} = c\ |q_{n+1}|^{-\frac{n}{p}}$. Hence the conformally equivalent form (4.1.8) follows from Theorem 4.1.1.

□

4.2 Sharp Constants for Multilinear Geometric Inequalities

We now turn to study the optimiser for multilinear inequality (4.2.1) as follows.

Theorem 4.2.1. *Let $0 < p < \infty$ and f_j be nonnegative measurable functions in $L^p(\mathbb{R}^n)$. For multilinear geometric inequality*

$$\prod_{j=1}^{n+1} \|f_j\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{\gamma} \tag{4.2.1}$$

with $\gamma = \frac{n+1}{p}$, the minimum constant is obtained when $f_j = const \cdot h$, $1 \leq j \leq n+1$, where

$$h(x) = (1 + |x|^2)^{-\frac{n+1}{2p}}.$$

Later we can see the sharp constant $C_{p,n} = (\frac{1}{2}|\mathbb{S}^n|)^{\frac{n+1}{p}}$, where $|\mathbb{S}^n|$ is the surface area of the unit sphere $\mathbb{S}^n$.

As before, it suffices to study the extremals for the case when $1 < p < \infty$. Because for any $q \in (0, \infty)$, $p \in (1, \infty)$, we have

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j)^{\frac{p}{q}} \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{q}} = (\sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}})^{\frac{p}{q}}.$$

and for each j,

$$\|f_j^{\frac{p}{q}}\|_{L^q(\mathbb{R}^n)} = (\|f_j\|_{L^p(\mathbb{R}^n)})^{\frac{p}{q}}$$

Thus if $\{f_j\}$ are the extremal functions for $p \in (1, \infty)$, then $\{f_j^{\frac{p}{q}}\}$ are the extremal functions for any $q \in (0, \infty)$. From Theorem 3.3.4, it suffices to seek optimisers amogst the class of all symmetric decreasing functions.

Let $\mathcal{S}$ be the stereographic projection from $\mathbb{R}^n$ to the nothern hemisphere $\mathbb{S}^n_+$ with

$$\mathcal{S}(x) = (\frac{x_1}{\sqrt{1+|x|^2}}, \ldots, \frac{x_n}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}}),$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. So

$$\mathcal{S}^{-1}(s) = (\frac{s_1}{s_{n+1}}, \ldots, \frac{s_n}{s_{n+1}}).$$

For $f \in L^p(\mathbb{R}^n)$, define

$$(\mathcal{S}^* f)(s) := |J_{\mathcal{S}^{-1}}(s)|^{1/p} f(\mathcal{S}^{-1}(s)) \tag{4.2.2}$$

where $J_{\mathcal{S}^{-1}}$ is the Jacobian determinant of the map $\mathcal{S}^{-1}$,

$$|J_{\mathcal{S}^{-1}}(s)| = (\frac{1}{s_{n+1}})^{n+1} = (1 + |\mathcal{S}^{-1}(s)|^2)^{\frac{n+1}{2}}. \tag{4.2.3}$$

Together with (4.2.2) and (4.2.3), we also have the conformally invariant property of multilinear geometric inequality (4.2.1) as follows.

Lemma 4.2.2. *Let $f_j \in L^p(\mathbb{R}^n)$ be nonnegative measurable functions, $1 \le j \le n+1$. Denote $\mathcal{S}^* f_j$ by F_j. Then*

$$\sup_{s_j \in \mathbb{S}^n_+} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, \dots, s_{n+1})^{\frac{n+1}{p}} = \sup_{y_j \in \mathbb{R}^n} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}},$$

where $\det(s_1, \dots, s_{n+1})$ is the absolute value of the determinant of the matrix $(s_1, \dots, s_{n+1})_{(n+1)\times(n+1)}$ and for each j

$$\|f_j\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{S}^n_+} |F_j(s_j)|^p ds_j)^{1/p}.$$

Proof. Let $y_j = \mathcal{S}^{-1}(s_j)$. From the proof of Theorem 8 in [47], we have (4.2.3). The stereographic projection $\mathcal{S}$ is

$$|J_{\mathcal{S}^{-1}}(s)| = (1 + |\mathcal{S}^{-1}(s)|^2)^{\frac{n+1}{2}},$$

and

$$\begin{aligned} \det(y_1, \dots, y_{n+1}) &= \prod_{j=1}^{n+1} (1 + |y_j|^2)^{1/2} \det(s_1, \dots, s_{n+1}) \\ &= \prod_{j=1}^{n+1} |J_{\mathcal{S}^{-1}}(s_j)|^{\frac{1}{n+1}} \det(s_1, \dots, s_{n+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{s_j \in \mathbb{S}^n_+} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, \dots, s_{n+1})^{\frac{n+1}{p}} &= \sup_{s_j \in \mathbb{S}^n_+} |J_{\mathcal{S}^{-1}}(s_j)|^{1/p} f_j(\mathcal{S}^{-1}(s_j)) \det(s_1, \dots, s_{n+1})^{\frac{n+1}{p}} \\ &= \sup_{y_j \in \mathbb{R}^n} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}}. \end{aligned}$$

and the invariance of L^p norm: for every j

$$\begin{aligned} \|f_j\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p} &= (\int_{\mathbb{S}^n_+} |f(\mathcal{S}^{-1}(s_j))|^p |J_{\mathcal{S}^{-1}}(s_j)| ds_j)^{1/p} \\ &= (\int_{\mathbb{S}^n_+} |F_j(s_j)|^p ds_j)^{1/p}. \end{aligned}$$

□

Together with Theorem 3.3.4, Lemma 4.2.2, it is not hard to see that taking the same arguments as in the proof of Theorem 4.1.1 gives the sharp constant of (4.2.1). That is, by competing symmetries if we construct the same sequence

$\{f_k\}_{k\in\mathbb{N}}$ as in the proof of Theorem 4.1.1,

$$f_0 = f,\ f_k = (\mathcal{R}\mathcal{D})^k f,$$

similar to the proof of Theorem 4.1.1 we have for all $f \in L^p(\mathbb{R}^n)$, the sequence f_k converges to h_f in L^p norm as $k \to \infty$. Here

$$h_f = c\, h,\quad h(x) = (1+|x|^2)^{-\frac{n+1}{2p}}$$

and c is the constant such that $\|f\|_p = \|h_f\|_p$. It follows from Theorem 3.3.4 together with Lemma 4.2.2 that $\sup_{y_j} \prod_{j=1}^{n+1} f_{j_k}(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}$ decreases monotonically as k grows. Apply the rest similar arguments in the proof of Theorem 4.1.1 to obtain that h is the optimiser for (4.2.1).

Here we present an alternative sequence $\{f^k\}_{k\in\mathbb{N}}$ to deduce that for all $f \in L^p(\mathbb{R}^n)$, the sequence f^k converges to h_f in L^p norm as well, mainly using the competing symmetries in one dimension togehter with Corollary 3.2.6 to get the optimisers in high dimension. This method has been applied in [11] to prove the extremals of the Hardy-Littlewood-Sobolev inequality and in [47] to prove the extremals of the multilinear determinant fractional integration .

Proof of Theorem 4.2.1

For $f \in L^p(\mathbb{R}^n)$, pick α which is not a rational multiple of π. For $1 \le i \le n$, we define $U^i_\alpha : \mathbb{S}^n_+ \to \mathbb{S}^n_+$ be a rotation of the sphere $\mathbb{S}^n$ by angle α which keeps the other basis vectors fixed except the i-th and $(n+1)$-th vectors. If the point after rotation is in the southern hemisphere, we then send the point to its antipodal point in $\mathbb{S}^n_+$. For $F \in L^p(\mathbb{S}^n_+)$, define

$$((U^i_\alpha)^* F)(s) := |J_{(U^i_\alpha)^{-1}}(s)|^{\frac{1}{p}} F((U^i_\alpha)^{-1}s) = F((U^i_\alpha)^{-1}s).$$

With the same $\mathcal{S}^*$ in (4.2.2), we consider the new function $(\mathcal{S}^*)^{-1}(U^i_\alpha)^*\mathcal{S}^* f$.

In brief we denote this new function $(\mathcal{S}^*)^{-1}(U^i_\alpha)^*\mathcal{S}^* f$ by $\mathcal{U}^i_\alpha f$. For any $f \in L^p(\mathbb{R}^n)$, we define a sequence $\{f^k\}$ as in [47] as follows,

$$f^0 = f,\ f^1 = \mathcal{R}_n\mathcal{R}_{n-1}\ldots\mathcal{R}_1\mathcal{U}^1_\alpha f,\ f^2 = \mathcal{R}_1\mathcal{R}_n\ldots\mathcal{R}_2\mathcal{U}^2_\alpha f^1,$$

$$f^3 = \mathcal{R}_2\mathcal{R}_1\mathcal{R}_n\ldots\mathcal{R}_3\mathcal{U}^3_\alpha f^2, \cdots, f^{n+1} = \mathcal{R}_n\ldots\mathcal{R}_1\mathcal{U}^1_\alpha f^n \cdots$$

Note that $\mathcal{U}^i_\alpha$ and $\mathcal{R}_n\mathcal{R}_{n-1}\ldots\mathcal{R}_1$ are norm-preserving. It follows from the proof of Theorem 8 in [47] that for any $f \in L^p(\mathbb{R}^n)$, we have $\{f^k\}$ converges to h_f in L^p norm, where

$$h_f = ch,\qquad h(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{n+1}{2p}}$$

and c is the constant such that $\|f\|_p = \|ch\|_p$. So

$$c = |\mathbb{S}^n_+|^{-\frac{1}{p}}\|f\|_p.$$

Here we will only sketch the argument, mainly using the competing symmetries in one dimension. First it is enough to consider the bounded functions that vanish outside a bounded set which are dense in L^p, so there exists a constant C such that $f(x) \leq Ch_f(x)$. Note that $\mathcal{R}_j$ and $\mathcal{U}_\alpha^j$ are order-preserving, then we have $f^k(x) \leq Ch_f(x)$ for every f^k and all x. By Helly's selection principle we can find a subsequence f^{k_l} such that f^{k_l} converges to some g almost everywhere as $l \to \infty$. The dominated convergence theorem implies that $g \in L^p$. We define

$$A := \inf_k \|h_f - f^k\|_p = \lim_{k\to\infty} \|h_f - f^k\|_p,$$

this is because $\|h_f - f^k\|_p$ decreases monotonically which follows from the property

$$\|\mathcal{R}_j f - \mathcal{R}_j g\|_p \leq \|f-g\|_p, \quad \|\mathcal{U}_\alpha^j f - \mathcal{U}_\alpha^j g\|_p = \|f-g\|_p \tag{4.2.4}$$

and the invariance of h_f under each $\mathcal{R}_j$ and $\mathcal{U}_\alpha^j$.
Applying these properties again gives that

$$\begin{aligned} A &= \lim_l \|h_f - f^{k_l+1}\|_p \\ &= \|h_f - \mathcal{R}_n\mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 g\|_p \\ &= \|\mathcal{R}_n\mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 h_f - \mathcal{R}_n\mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 g\|_p \\ &\leq \|\mathcal{U}_\alpha^1 h_f - \mathcal{U}_\alpha^1 g\|_p \\ &= \|h_f - g\|_p = A, \end{aligned} \tag{4.2.5}$$

then we must have equality everywhere

$$\begin{aligned} \|h_f - \mathcal{R}_n\mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 g\|_p &= \|h_f - \mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 g\|_p \\ &= \dots \\ &= \|h_f - \mathcal{R}_1\mathcal{U}_\alpha^1 g\|_p \\ &= \|h_f - \mathcal{U}_\alpha^1 g\|_p \end{aligned} \tag{4.2.6}$$

which implies (see Proposition 3.1.1 nonexpansivity of rearrangement)

$$\mathcal{R}_n\mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 g = \mathcal{R}_{n-1}\dots\mathcal{R}_1\mathcal{U}_\alpha^1 g = \dots = \mathcal{R}_1\mathcal{U}_\alpha^1 g = \mathcal{U}_\alpha^1 g \tag{4.2.7}$$

It turns out that $\mathcal{R}_1\mathcal{U}_\alpha^1 g = \mathcal{U}_\alpha^1 g$ and $\mathcal{R}_1 g = g$ imply $\mathcal{U}_{2\alpha}^1 g = g$ which shows $\mathcal{S}^* g$ is invariant under the rotation through an angle 2α which keeps the other basis vectors fixed except the 1-th and $(n+1)$-th ones. In particular, 2α is an irrational multiple of π. Therefore, for any fixed $s_2, \dots, s_n$, $(\mathcal{S}^* g)(\cdot, s_2, \dots, s_n, \cdot)$ is a constant. Also we have

$$\mathcal{R}_1\mathcal{U}_\alpha^1 g = \mathcal{U}_\alpha^1 g = g. \tag{4.2.8}$$

Similarly, if we replace f^{k_l+1} in (4.2.5) by f^{k_l+2}, together with (4.2.7)-(4.2.8) and Proposition 3.1.1 we have

$$\mathcal{R}_1\mathcal{R}_n\dots\mathcal{R}_2\mathcal{U}_\alpha^2 g = \mathcal{R}_n\dots\mathcal{R}_2\mathcal{U}_\alpha^2 g = \dots = \mathcal{R}_2\mathcal{U}_\alpha^2 g = \mathcal{U}_\alpha^2 g. \tag{4.2.9}$$

From $\mathcal{R}_2\mathcal{U}_\alpha^2 g = \mathcal{U}_\alpha^2 g$ and $\mathcal{R}_2 g = g$ we obtain that $\mathcal{U}_{2\alpha}^2 g = g$ which shows $\mathcal{S}^* g$

is invariant under the rotation through an angle 2α which keeps the other basis vectors fixed except the 2-th and $(n+1)$-th ones. So for any fixed $s_1, s_3, \dots, s_n$, $(\mathcal{S}^* g)(s_1, \cdot, s_3, \dots, s_n, \cdot)$ is a constant, since 2α is an irrational multiple of π. Meanwhile we have

$$\mathcal{R}_2 \mathcal{U}_\alpha^2 g = \mathcal{U}_\alpha^2 g = g. \tag{4.2.10}$$

So far based on the discussion above, we've got for any fixed $s_3, \dots, s_n$, $(\mathcal{S}^* g)(\cdot, \cdot, s_3, \dots, s_n, \cdot)$ must be a constant.

By induction we can obtain $\mathcal{S}^* g$ is a constant function on $\mathbb{S}^n_+$, and thus the corresponding function g on $\mathbb{R}^n$ is Ch_f. Note that $\mathcal{R}_j$ and $\mathcal{U}_\alpha^j$ are norm-preserving, so

$$\|g\|_p = \lim_l \|f^{k_l}\|_p = \|f\|_p,$$

which gives $C = 1$, $g = h_f$. Therefore, the sequence f^k converges to h_f in L^p norm. Lastly, it follows from Theorem 3.2.5, Lemma 4.2.2, Corollary 3.2.6 and the rearrangement property for $0 < p < \infty$

$$(f^p)^* = (f^*)^p$$

that $\sup_{y_j} \prod_{j=1}^{n+1} f_j^k(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}}$ decreases monotonically as k grows. That is for all $k \in \mathbb{N}$,

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^k(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}} \geq \sup_{y_j} \prod_{j=1}^{n+1} f_j^{k+1}(y_j) \ \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}}.$$

Since $\{f_j^k\}$ converges to h_{f_j} in L^p norm , $1 \leq j \leq n+1$, then there exist subsequences $\{f_j^{k_l}\}$ such that $f_j^{k_l} \to h_{f_j}$ pointwise almost everywhere as $l \to \infty$. From

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^{k_l}(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}} \leq \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \ \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}} < \infty$$

for all k_l together with the dominated convergence theorem it follows that

$$\prod_{j=1}^{n+1} f_j^{k_l}(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}} \xrightarrow{weak^*} \prod_{j=1}^{n+1} h_{f_j}(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}},$$

in $L^\infty(\mathbb{R}^n) \times \cdots \times L^\infty(\mathbb{R}^n)$ as $l \to \infty$.
Hence by the weak* lower semicontinuity of the L^∞ norm,

$$\begin{aligned}
\sup_{y_j} \prod_{j=1}^{n+1} h_{f_j}(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}} &\leq \liminf_{l \to \infty} (\sup_{y_j} \prod_{j=1}^{n+1} f_j^{k_l}(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}} \\
&= \inf_l (\sup_{y_j} \prod_{j=1}^{n+1} f_j^{k_l}(y_j) \det(y_1, \dots, y_{n+1})^{\frac{n+1}{p}}.
\end{aligned}$$

Combining this with the norm-preserving property $\|f_j\|_p = \|f_j^k\|_p$ for every k and the decreasing property of $\sup_{y_j} \prod_{j=1}^{n+1} f_j^k(y_j)\ \det(y_1, ..., y_{n+1})^{\frac{n+1}{p}}$, we get for all $f_j \in L^p(\mathbb{R}^n)$ and every k_l,

$$\frac{\sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}}{\prod_{j=1}^{n+1} \|f_j\|_p} \geq \frac{\sup_{y_j} \prod_{j=1}^{n+1} f_j^{k_l}(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}}{\prod_{j=1}^{n+1} \|f_j^{k_l}\|_p}$$

$$\geq \frac{\sup_{y_j} \prod_{j=1}^{n+1} h_{f_j}(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}}{\prod_{j=1}^{n+1} \|h_{f_j}\|_p}.$$

Obviously,

$$\frac{\sup_{y_j} \prod_{j=1}^{n+1} h_{f_j}(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}}{\prod_{j=1}^{n+1} \|h_{f_j}\|_p} = \frac{\sup_{y_j} \prod_{j=1}^{n+1} h(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}}{\|h\|_p^{n+1}}.$$

□

Based on Theorem 4.2.1 and the conformal invariance under the stereographic projection from $\mathbb{R}^n$ to $\mathbb{S}^n_+$, the geometric inequality (4.2.1) has the conformally equivalent form in $\mathbb{S}^n$ space.

Theorem 4.2.3. *For $0 < p < \infty$, let F_j be nonnegative functions in $L^p(\mathbb{S}^n)$. Then*

$$\prod_{j=1}^{n+1} \|F_j\|_{L^p(\mathbb{S}^n)} \leq B_{p,n} \sup_{s_j \in \mathbb{S}^n} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}}, \tag{4.2.11}$$

where $\det(s_1, \ldots, s_{n+1})$ is the absolute value of the determinant of the matrix $(s_1, \ldots, s_{n+1})_{(n+1)\times(n+1)}$. The best constant $B_{p,n}$ is obtained when $F_j(s_j)$ are constant, and the corresponding $B_{p,n} = |\mathbb{S}^n|^{\frac{n+1}{p}}$.

Proof. From Theorem 4.2.1 and and the conformal invariance of (4.2.1) under the stereographic projection from $\mathbb{R}^n$ to $\mathbb{S}^n_+$, we obtain for nonnegative functions $F_j \in L^p(\mathbb{S}^n_+)$,

$$\prod_{j=1}^{n+1} \|F_j\|_{L^p(\mathbb{S}^n_+)} \leq C_{p,n} \sup_{s_j \in \mathbb{S}^n_+} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, ..., s_{n+1})^{\frac{n+1}{p}} \tag{4.2.12}$$

holds. The best constant $C_{p,n}$ is obtained when $F_j(s_j)$ are constant, and the corresponding $C_{p,n} = |\mathbb{S}^n_+|^{\frac{n+1}{p}} = (\frac{1}{2}|\mathbb{S}^n|)^{\frac{n+1}{p}}$. Note that

$$\sup_{s_j \in \mathbb{S}^n_+} \det(s_1, \ldots, s_{n+1}) = \sup_{s_j \in \mathbb{S}^n} \det(s_1, \ldots, s_{n+1}) = 1.$$

Let F_j be nonnegative functions in $L^p(\mathbb{S}^n)$. We define

$$\overline{F}_j(s_j) = \max\{F_j(s_j), F_j(\overline{s}_j)\},$$

where $\overline{s}_j$ is the antipodal point of s_j, $1 \leq j \leq n+1$, $s_j \in \mathbb{S}^n_+$.
Then $\overline{F}_j \in L^p(\mathbb{S}^n_+)$, and

$$\sup_{s_j \in \mathbb{S}^n_+} \prod_{j=1}^{n+1} \overline{F}_j(s_j) \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}} = \sup_{s_j \in \mathbb{S}^n} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}}, \tag{4.2.13}$$

this is because for any $s_j \in \mathbb{S}^n_+$,

$$\det(s_1, \ldots, \overline{s}_j, \ldots, s_{n+1}) = \det(s_1, \ldots, s_j, \ldots, s_{n+1}).$$

Besides,

$$2\|\overline{F}_j\|^p_{L^p(\mathbb{S}^n_+)} \geq \int_{\mathbb{S}^n_+} (F_j(s_j))^p ds_j + \int_{\mathbb{S}^n_+} (F_j(\overline{s}_j))^p ds_j = \|F_j\|^p_{L^p(\mathbb{S}^n)}.$$

Thus for each j

$$\|\overline{F}_j\|_{L^p(\mathbb{S}^n_+)} \geq 2^{-\frac{1}{p}} \|F_j\|_{L^p(\mathbb{S}^n)}. \tag{4.2.14}$$

It follows from (4.2.12)-(4.2.14) that for any nonnegative $F_j \in L^p(\mathbb{S}^n)$,

$$\begin{aligned}
\prod_{j=1}^{n+1} \|F_j\|_{L^p(\mathbb{S}^n)} &\leq 2^{\frac{n+1}{p}} \prod_{j=1}^{n+1} \|\overline{F}_j\|_{L^p(\mathbb{S}^n_+)} \\
&\leq 2^{\frac{n+1}{p}} |\mathbb{S}^n_+|^{\frac{n+1}{p}} \sup_{s_j \in \mathbb{S}^n_+} \prod_{j=1}^{n+1} \overline{F}(s_j) \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}} \\
&= |\mathbb{S}^n|^{\frac{n+1}{p}} \sup_{s_j \in \mathbb{S}^n} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}},
\end{aligned}$$

which proves (4.2.11).

To show that $|\mathbb{S}^n|^{\frac{n+1}{p}}$ is the best constant in (4.2.11), suppose for a contradiction that $F_j(s_j) \in L^p(\mathbb{S}^n)$ is an optimiser for (4.2.11) that satisfies

$$\frac{\sup_{s_j \in \mathbb{S}^n} \prod_{j=1}^{n+1} F_j(s_j) \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}}}{\prod_{j=1}^{n+1} \|F_j\|_{L^p(\mathbb{S}^n)}} < \frac{\sup_{s_j \in \mathbb{S}^n} \det(s_1, \ldots, s_{n+1})^{\frac{n+1}{p}}}{|\mathbb{S}^n|^{\frac{1}{p}} \cdots |\mathbb{S}^n|^{\frac{1}{p}}} = |\mathbb{S}^n|^{-\frac{n+1}{p}}.$$

Then by (4.2.13)-(4.2.14) we find $\overline{F}_j(s_j)$ defined as above satisfying

$$\frac{\sup_{s_j\in\mathbb{S}^n_+}\prod_{j=1}^{n+1}\overline{F}_j(s_j)\det(s_1,\ldots,s_{n+1})^{\frac{n+1}{p}}}{\prod_{j=1}^{n+1}\|\overline{F}_j\|_{L^p(\mathbb{S}^n_+)}} \le \frac{\sup_{s_j\in\mathbb{S}^n}\prod_{j=1}^{n+1}F_j(s_j)\det(s_1,\ldots,s_{n+1})^{\frac{n+1}{p}}}{2^{-\frac{n+1}{p}}\prod_{j=1}^{n+1}\|F_j\|_{L^p(\mathbb{S}^n)}}$$
$$< 2^{\frac{n+1}{p}}|\mathbb{S}^n|^{-\frac{n+1}{p}} = |\mathbb{S}^n_+|^{-\frac{n+1}{p}}.$$

This is in contradiction to the best constant in (4.2.12). Hence the best constant $B_{p,n}$ in (4.2.11) is $|\mathbb{S}^n|^{\frac{n+1}{p}}$.

□

Also the multilinear geometric inequality (4.2.1) has a version in $\mathbb{H}^n$ space as shown in the following theorem.

Theorem 4.2.4. *For $0 < p < \infty$, let F_j be nonnegative functions in $L^p(\mathbb{H}^n)$. Then*

$$\prod_{j=1}^{n+1}\|F_j\|_{L^p(\mathbb{H}^n)} \le E_{p,n}\sup_{q_j\in\mathbb{H}^n}\prod_{j=1}^{n+1}F_j(q_j)\det(q_1,\ldots,q_{n+1})^{\frac{n+1}{p}}, \tag{4.2.15}$$

where $\det(q_1,\ldots,q_{n+1})$ *is the absolute value of the determinant of the matrix* $(q_1,\ldots,q_{n+1})_{(n+1)\times(n+1)}$.

Proof. Consider the stereographic projection $\mathcal{H}$ which is a conformal transformation from the unit disk D^n in $\mathbb{R}^n$ to $\mathbb{H}^n_+$ as

$$\mathcal{H}(x) = \left(\frac{x_1}{\sqrt{1-|x|^2}},\ldots,\frac{x_n}{\sqrt{1-|x|^2}},\frac{1}{\sqrt{1-|x|^2}}\right),$$

where $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$. So

$$\mathcal{H}^{-1}(q) = \left(\frac{q_1}{q_{n+1}},\ldots,\frac{q_n}{q_{n+1}}\right).$$

Let $y_j = \mathcal{H}^{-1}(q_j)$. It follows from [47] that the Jacobian determinant of the map $\mathcal{H}^{-1}$ is

$$|J_{\mathcal{H}^{-1}(q)}| = (1-|\mathcal{H}^{-1}(q)|^2)^{\frac{n+1}{2}},$$

and

$$\det(y_1,\ldots,y_{n+1}) = \prod_{j=1}^{n+1}(1-|y_j|^2)^{1/2}\det(q_1,\ldots,q_{n+1})$$
$$= \prod_{j=1}^{n+1}|J_{\mathcal{H}^{-1}}(q_j)|^{\frac{1}{n+1}}\det(q_1,\ldots,q_{n+1}).$$

Define $F_j(q_j) := |J_{\mathcal{H}^{-1}}(q_j)|^{1/p} f_j(\mathcal{H}^{-1}(q_j))$, then from above we easily get the conformal invariance as follows.

$$
\begin{aligned}
&\sup_{q_j \in \mathbb{H}^n_+} \prod_{j=1}^{n+1} F_j(q_j) \det(q_1, \ldots, q_{n+1})^{\frac{n+1}{p}} \\
= &\sup_{q_j \in \mathbb{H}^n_+} \prod_{j=1}^{n+1} |J_{\mathcal{H}^{-1}}(q_j)|^{1/p} f_j(\mathcal{H}^{-1}(q_j)) \det(q_1, \ldots, q_{n+1})^{\frac{n+1}{p}} \\
= &\sup_{y_j \in D^n} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}},
\end{aligned}
$$

and for every j

$$
\begin{aligned}
\|f_j\|_{L^p(D^n)} &= (\int_{D^n} |f_j(y_j)|^p dy_j)^{1/p} \\
&= (\int_{\mathbb{H}^n_+} |f_j(\mathcal{H}^{-1}(q_j))|^p |J_{\mathcal{H}^{-1}(q)}| dq_j)^{1/p} \\
&= (\int_{\mathbb{H}^n_+} |F_j(q_j)|^p dq_j)^{1/p} = \|F_j\|_{L^p(\mathbb{H}^n_+)}.
\end{aligned}
$$

Theorem 4.2.1 implies that

$$
\prod_{j=1}^{n+1} \|f_j\|_{L^p(D^n)} \leq C_{p,n} \sup_{y_j \in D^n} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{\frac{n+1}{p}}, \tag{4.2.16}
$$

where $C_{p,n} = |\mathbb{S}^n_+|^{\frac{n+1}{p}}$. Thus from the discussion above we have

$$
\prod_{j=1}^{n+1} \|F_j\|_{L^p(\mathbb{H}^n_+)} \leq |\mathbb{S}^n_+|^{\frac{n+1}{p}} \sup_{q_j \in \mathbb{H}^n_+} \prod_{j=1}^{n+1} F_j(q_j) \det(q_1, \ldots, q_{n+1})^{\frac{n+1}{p}}.
$$

Similarly to the arguments in the proof of Theorem 4.2.3, we also have for any nonnegative $F_j \in L^p(\mathbb{H}^n)$, $0 < p < \infty$, there exists a finite constant $E_{p,n} = 2^{\frac{n+1}{p}} |\mathbb{S}^n_+|^{\frac{n+1}{p}}$ such that

$$
\prod_{j=1}^{n+1} \|F_j\|_{L^p(\mathbb{H}^n)} \leq E_{p,n} \sup_{q_j \in \mathbb{H}^n} \prod_{j=1}^{n+1} F_j(q_j) \det(q_1, \ldots, q_{n+1})^{\frac{n+1}{p}}.
$$

□

Remark 4.2.5. For the multilinear determinant form, the stereographic projection we give in Theorem 4.2.4 is from D^n to $\mathbb{H}^n$. Moreover, we do not know the extremal functions and the best constant $C_{p,n}$ for inequality (4.2.16). So a problem is to determine the optimisers and the best constant $E_{p,n}$ for the multilinear geometric inequality (4.2.15) in hyperbolic space $\mathbb{H}^n$, and if (4.2.1) has conformally equivalent form on $\mathbb{H}^n$. However, (4.1.1) has conformally equivalent form on $\mathbb{H}^n$ as shown in Theorem 4.1.4. For the bilinear form, there is a stereographic projection from $\mathbb{R}^n$ to $\mathbb{H}^n$. Let $x \in \mathbb{R}^n$. Draw a straight line in $\mathbb{R}^{n+1}$ through $(x, 0)$ and $(0, \ldots, 0, -1)$. The stereographic projection maps x to the intersection point with $\mathbb{H}^n$. It is easy to see it is a conformal transformation from $\mathbb{R}^n$ to $\mathbb{H}^n$. When $|x| < 1$, the intersection point with $\mathbb{H}^n$ is in the upper half of the hyperbolic space; when $|x| > 1$, the intersection point is in the lower half of the hyperbolic space.

Chapter 5

The Regularity of Hardy-Littlewood Maximal Functions

In this chapter we present some results in a joint paper with F. Liu and H. Wu [35], mainly studying the regularity of Hardy-Littlewood maximal functions at the endpoint case in one dimension.

5.1 A Brief History of Regularity Problems

Definition 5.1.1. *For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the function*

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

is called the centered Hardy-Littlewood maximal function of f.

As is well known, the operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ for any $1 < p \leq \infty$ and maps $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Definition 5.1.2. *The Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $1 \leq p \leq \infty$, are defined by*

$$W^{1,p}(\mathbb{R}^d) := \{f : \ \|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla(f)\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))$ is the weak gradient.

In 1997, Kinnunen [27] first studied the regularity of $\mathcal{M}$ and showed that $\mathcal{M}$ is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for all $1 < p \leq \infty$. Subsequently, Kinnunen and Lindqvist [28] gave a local version of the original boundedness on $W^{1,p}(\Omega)$, where Ω is an open set of $\mathbb{R}^n$. Later on, the continuity of $\mathcal{M} : W^{1,p} \to W^{1,p}$ for $p > 1$ was established by Luiro in [36] and in [37] for its local version (continuity is not immediate from boundedness because of the lack of linearity).

As usual, the endpoint case $p = 1$ is significantly different from the case $p > 1$, not only because $\mathcal{M}f \notin L^1(\mathbb{R}^n)$ whenever f is nontrival, while the maxiaml operator acts boundedly on L^p for $p > 1$, but also because $L^1(\mathbb{R}^n)$ is not reflexive so weak compactness arguments used when $1 < p < \infty$ are not available for $p = 1$. Since Kinnunen's result doesn't hold for the case $p = 1$, understanding the regularity at the endpoint case seems to be a deeper issue. In 2004, Hajlasz and Onninen [23] asked the following question:

Question. Is the operator $f \mapsto |\nabla \mathcal{M}(f)|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?

In Tanaka's paper [46], he gave a positive answer to this question for the non-centred Hardy-Littlewood maximal function in dimension $n = 1$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the non-centred Hardy-Littlewood maximal function is defined by

$$\widetilde{\mathcal{M}}f(x) = \sup_{s,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(y)|dy.$$

He showed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{\mathcal{M}}f$ is weakly differentiable on $\mathbb{R}$ and

$$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \leq 2\|f'\|_{L^1(\mathbb{R})}. \tag{5.1.1}$$

The result was later refined by Aldaz and Pérez-Lázaro [2] who obtained, under the assumption that f is of bounded variation on $\mathbb{R}$, $\widetilde{\mathcal{M}}f$ is absolutely continuous and

$$\mathrm{Var}(\widetilde{\mathcal{M}}f) \leq \mathrm{Var}(f),$$

where $\mathrm{Var}(f)$ is the total variation of f. This implies if $f \in W^{1,1}(\mathbb{R})$, then

$$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \leq \|f'\|_{L^1(\mathbb{R})},$$

which is an improvement of Tanaka's result. For centred Hardy-Littlewood maximal function, under the same assumption that f is of bounded variation on $\mathbb{R}$ Kurka [31] recently showed that

$$\mathrm{Var}(\mathcal{M}f) \leq C\mathrm{Var}(f)$$

with certain constant $C > 1$.

In collaboration with F. Liu and H. Wu [35], we give a simple and elementary proof to improve Tanaka's result (5.1.1) under the same assumption $f \in W^{1,1}(\mathbb{R})$. Our proof mainly relies on the property of the local maximum of the non-centred Hardy-Littlewood maximal function. However, our method does not work for $\mathcal{M}$.

5.2 An Improvement of Tanaka's Result

The improvement result, Theorem 5.2.5, is presented in Subsection 5.2.2.

5.2.1 Identity on Local Maximal Points

In this section we study the behavior of the local maximum of non-centred Hardy-Littlewood maximal functions to obtain a important identity on the local maximum points shown in Theorem 5.2.3. We first introduce the definition of local maximum points as follows.

Definition 5.2.1. *We say that a point x_0 is a local maximum of f if there exists $\alpha > 0$ such that*

$$f(x_0) \geq f(x_0 - h) \text{ and } f(x_0) \geq f(x_0 + h), \ \forall\ 0 < h < \alpha.$$

Lemma 5.2.2. *If f is continuous and integrable on $\mathbb{R}$, then $\widetilde{\mathcal{M}}f$ is continuous on $\mathbb{R}$ and $\widetilde{\mathcal{M}}f(x) \geq |f(x)|$ for all $x \in \mathbb{R}$. Moreover, if $f \in W^{1,1}(\mathbb{R})$, then both f and $\widetilde{\mathcal{M}}(f)$ vanish at infinity.*

Proof. It follows from Lebesgue differentiation theorem that

$$\widetilde{\mathcal{M}}f(x) \geq |f(x)|, \quad a.e.\ x \in \mathbb{R}.$$

If f is continuous on $\mathbb{R}$, Lebesgue differentiation theorem holds for all $x \in \mathbb{R}$, not just almost all x on $\mathbb{R}$. So we have

$$\widetilde{\mathcal{M}}f(x) \geq |f(x)|, \quad \forall\ x \in \mathbb{R}.$$

Below we prove the continuity of $\widetilde{\mathcal{M}}(f)$. When $\|f\|_{L^1(\mathbb{R})} = 0$, by the continuity of f we have $f \equiv 0$ and the conclusions are obvious. When $\|f\|_{L^1(\mathbb{R})} \neq 0$, for any x, $h \in \mathbb{R}$, one can easily check that

$$|\widetilde{\mathcal{M}}f(x+h) - \widetilde{\mathcal{M}}f(x)| \leq \sup_{s,\,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(y+h) - f(y)| dy.$$

For any $\epsilon > 0$, we set $\delta_1 = 2\|f\|_{L^1(\mathbb{R})}/\epsilon$. Since f is uniformly continuous on $[x-2\delta_1, x+2\delta_1]$, for any $\epsilon > 0$ there exists positive $\delta < \delta_1$ such that $|f(y)-f(z)| < \epsilon$ for all $y, z \in [x - 2\delta_1, x + 2\delta_1]$ with $|y - z| < \delta$. Then we consider the following two cases:

(i) If $s + t \leq \delta_1$, for all $|h| < \delta$

$$\frac{1}{s+t} \int_{x-s}^{x+t} |f(y+h) - f(y)| dy \leq \frac{1}{s+t} \int_{x-s}^{x+t} \epsilon\ dy < \epsilon,$$

which is due to the fact that $[x - s, x + t] \subset [x - \delta_1, x + \delta_1]$ and f is uniformly continuous on $[x - 2\delta_1, x + 2\delta_1]$.

(ii) If $s + t > \delta_1$, then for all h

$$\frac{1}{s+t} \int_{x-s}^{x+t} |f(y+h) - f(y)| dy \leq \frac{2}{s+t} \|f\|_{L^1(\mathbb{R})} < \epsilon.$$

Thus we have for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $|h| < \delta$

$$|\widetilde{\mathcal{M}}f(x+h) - \widetilde{\mathcal{M}}f(x)| < \epsilon.$$

The continuity of $\widetilde{\mathcal{M}}f$ follows from this.

Moreover, if $f \in W^{1,1}(\mathbb{R})$, then f is absolutely continuous on $\mathbb{R}$, and its classical derivative is equal to the weak derivative almost everywhere. It follows from the fundamental theorem of calculus that for any $x \in \mathbb{R}$

$$f(x) - f(0) = \int_0^x f'(t)dt.$$

Taking $x \to \infty$, by dominated convergence theorem

$$\lim_{x\to\infty} f(x) = \int_0^\infty f'(t)dt + f(0).$$

Let $\lim\limits_{x\to\infty} f(x) = \alpha$, then $\alpha = 0$ since f is integrable on $\mathbb{R}$. Similarly, we can obtain that $\lim\limits_{x\to-\infty} f(x) = 0$. Thus f vanishes at infinity.

We shall claim that $\widetilde{\mathcal{M}}f$ vanishes at infinity. In fact, when $\|f\|_{L^1(\mathbb{R})} = 0$, then we have $f \equiv 0$ and the claim is trivial. When $\|f\|_{L^1(\mathbb{R})} \neq 0$, since f vanishes at infinity, then for any $\epsilon > 0$, there exists $B_1 > 0$ such that $|f(x)| < \epsilon$ for all $|x| > B_1$. Let $B_2 = B_1 + \|f\|_{L^1(\mathbb{R})}/\epsilon$. Then for any $|x| > B_2$, we consider the following two cases:

(1) If one of that $s > \|f\|_{L^1(\mathbb{R})}/\epsilon$ and $t > \|f\|_{L^1(\mathbb{R})}/\epsilon$ holds, Without loss of generality we may assume that $s > \|f\|_{L^1(\mathbb{R})}/\epsilon$, then

$$\frac{1}{s+t}\int_{x-s}^{x+t} |f(y)|dy < \frac{1}{s}\|f\|_{L^1(\mathbb{R})} < \epsilon.$$

(2) If $0 < s \leq \|f\|_{L^1(\mathbb{R})}/\epsilon$ and $0 < t \leq \|f\|_{L^1(\mathbb{R})}/\epsilon$, then we have $x+t > x-s > B_2 - s \geq B_1$ or $x - s < x + t < -B_2 + t \leq -B_1$. Consequently,

$$\frac{1}{s+t}\int_{x-s}^{x+t} |f(y)|dy < \frac{\epsilon(t+s)}{s+t} = \epsilon,$$

which concludes our claim.

□

Below is a main observation concerned with the local maximum points of the corresponding maximal function, which will play a key role in the proof of the main result.

Theorem 5.2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and integrable function. Suppose that x_0 is a local maximum of $\widetilde{\mathcal{M}}f$, then*

$$\widetilde{\mathcal{M}}f(x_0) = |f(x_0)|.$$

Proof. Since x_0 is a local maximum of $\widetilde{\mathcal{M}}f$, then there exists $\alpha > 0$ such that

$$\widetilde{\mathcal{M}}f(y) \leq \widetilde{\mathcal{M}}f(x_0), \quad \forall\ y \in (x_0 - \alpha, x_0 + \alpha).$$

We assume that $\widetilde{\mathcal{M}}f(x_0) > |f(x_0)|$ and set

$$\beta := \widetilde{\mathcal{M}}f(x_0) - |f(x_0)| > 0. \tag{5.2.1}$$

Since f is continuous, there exists $\delta > 0$ such that

$$|f(x)| < |f(x_0)| + \frac{\beta}{2}, \quad \forall\, |x - x_0| < \delta. \tag{5.2.2}$$

We consider the following two cases:

Case 1: Suppose $\widetilde{\mathcal{M}}f(x_0)$ is attained for some $s_0 \geq 0, t_0 \geq 0$ such that

$$\widetilde{\mathcal{M}}f(x_0) = \frac{1}{s_0 + t_0} \int_{x_0 - s_0}^{x_0 + t_0} |f(x)| dx. \tag{5.2.3}$$

Then we must have $s_0 + t_0 > 0$ because of (5.2.1)-(5.2.2).

If $s_0, t_0 > 0$, we choose a positive ε such that $\varepsilon < \min\{s_0, t_0, \alpha\}$, then

$$\begin{aligned}
\widetilde{\mathcal{M}}f(x_0) &= \frac{1}{s_0 + t_0} \int_{x_0 - s_0}^{x_0 + t_0} |f(x)| dx \\
&= \frac{s_0 - \varepsilon}{s_0 + t_0} \frac{1}{s_0 - \varepsilon} \int_{x_0 - s_0}^{x_0 - \varepsilon} |f(x)| dx + \frac{2\varepsilon}{s_0 + t_0} \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| dx \\
&\quad + \frac{t_0 - \varepsilon}{s_0 + t_0} \frac{1}{t_0 - \varepsilon} \int_{x_0 + \varepsilon}^{x_0 + t_0} |f(x)| dx \\
&\leq \frac{s_0 + t_0 - 2\varepsilon}{s_0 + t_0} \widetilde{\mathcal{M}}f(x_0) + \frac{2\varepsilon}{s_0 + t_0} \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| dx.
\end{aligned}$$

The last inequality follows since x_0 is a local maximum of $\widetilde{\mathcal{M}}f$. Therefore,

$$\frac{2\varepsilon}{s_0 + t_0} \widetilde{\mathcal{M}}f(x_0) \leq \frac{2\varepsilon}{s_0 + t_0} \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| dx.$$

That is,

$$\widetilde{\mathcal{M}}f(x_0) \leq \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| dx. \tag{5.2.4}$$

Let $\varepsilon \to 0$, applying Lebesgue differentiation theorem we deduce that

$$\widetilde{\mathcal{M}}f(x_0) \leq |f(x_0)|.$$

If $s_0 = 0$ or $t_0 = 0$, we can get $\widetilde{\mathcal{M}}f(x_0) \leq |f(x_0)|$ by a minor modification of the argument above. Combinging with Lemma 5.2.2, we obtain $\widetilde{\mathcal{M}}(f)(x_0) = |f(x_0)|$.

Case 2: Suppose that $\widetilde{\mathcal{M}}f(x_0)$ is not attained for any $s \geq 0$ or not attained for any $t \geq 0$. Without loss of generality we may assume that $\widetilde{\mathcal{M}}f(x_0)$ is not attained for any $s \geq 0$. Then we must have

$$\widetilde{\mathcal{M}}f(x_0) = \sup_{s > k, t > 0} \frac{1}{s + t} \int_{x_0 - s}^{x_0 + t} |f(x)| dx, \quad \forall\, k = 1, 2, \ldots. \tag{5.2.5}$$

Otherwise, there exists some $M > 0$ such that

$$\widetilde{\mathcal{M}}f(x_0) = \sup_{0<s\le M, t>0} \frac{1}{s+t} \int_{x_0-s}^{x_0+t} |f(x)|dx$$

which gives the contradiction to the assumption. Clearly,

$$\widetilde{\mathcal{M}}f(x_0) \le \sup_{s>k} \frac{1}{s} \|f\|_1, \quad \forall\ k = 1, 2, \dots, \tag{5.2.6}$$

which implies $\widetilde{\mathcal{M}}f(x_0) = 0$. Thus $\widetilde{\mathcal{M}}f(x_0) = |f(x_0)| = 0$. Sum up, we obtain that $\widetilde{\mathcal{M}}f(x_0) = |f(x_0)|$ and this completes the proof of Theorem 5.2.3.

□

5.2.2 Proof of the Main Result

Before stating the main result, we need the following lemma.

Lemma 5.2.4. *Let f be a function on $\mathbb{R}$. Let (a, b) be an interval such that both f and $\widetilde{\mathcal{M}}f$ are continuous on (a, b). Suppose that $\widetilde{\mathcal{M}}f(x) > |f(x)|$ for any $x \in (a, b)$, and $\widetilde{\mathcal{M}}f$ is strictly monotonic on (a, b). Then $\widetilde{\mathcal{M}}f$ is absolutely continuous on (a, b). Moreover, if one-sided limits $\lim_{x\to a^+} \widetilde{\mathcal{M}}f(x)$, $\lim_{x\to b^-} \widetilde{\mathcal{M}}f(x)$ exist, then $\widetilde{\mathcal{M}}f$ is Lipschitz on (a, b).*

Proof. Without loss of generality, we may assume $\widetilde{\mathcal{M}}(f)$ is strictly increasing on (a, b). Let

$$G := \{x \in (a, b) : (\widetilde{\mathcal{M}}f)'(x) = \infty\}.$$

In order to prove that $\widetilde{\mathcal{M}}f$ is absolutely continuous, since $\widetilde{\mathcal{M}}f$ is continutious and strictly increasing on (a, b), it suffices to show that $|\widetilde{\mathcal{M}}f(G)| = 0$. Below we shall prove $G = \emptyset$.

For any fixed $x_0 \in (a, b)$, since $\widetilde{\mathcal{M}}f(x_0) > |f(x_0)|$ and (5.2.1)-(5.2.2),

$$\widetilde{\mathcal{M}}f(x_0) \ne \sup_{0<s<\delta, 0<t<\delta} \frac{1}{s+t} \int_{x_0-s}^{x_0+t} |f(y)|dy$$

with the same δ in (5.2.2). Therefore, for any $\epsilon > 0$, there exist $s_0 > 0$ and $t_0 > 0$ such that $s_0 + t_0 > \delta$ and

$$\widetilde{\mathcal{M}}f(x_0) < \frac{1}{s_0+t_0} \int_{x_0-s_0}^{x_0+t_0} |f(y)|dy + \epsilon.$$

For any $h > 0$, we have

$$\begin{aligned}\widetilde{\mathcal{M}}f(x_0) - \widetilde{\mathcal{M}}f(x_0 - h) &\le \frac{1}{s_0 + t_0}\int_{x_0 - s_0}^{x_0 + t_0} |f(y)|dy + \epsilon - \frac{1}{s_0 + t_0 + h}\int_{x_0 - h - s_0}^{x_0 + t_0} |f(y)|dy \\ &\le \frac{1}{s_0 + t_0}\int_{x_0 - s_0}^{x_0 + t_0} |f(y)|dy + \epsilon - \frac{1}{s_0 + t_0 + h}\int_{x_0 - s_0}^{x_0 + t_0} |f(y)|dy \\ &= \frac{h}{(s_0 + t_0)(s_0 + t_0 + h)}\int_{x_0 - s_0}^{x_0 + t_0} |f(y)|dy + \epsilon.\end{aligned}$$

Since $s_0 + t_0 > \delta$,

$$\begin{aligned}\widetilde{\mathcal{M}}f(x_0) - \widetilde{\mathcal{M}}f(x_0 - h) &\le \frac{h}{\delta(s_0 + t_0)}\int_{x_0 - s_0}^{x_0 + t_0} |f(y)|dy + \epsilon \\ &\le \frac{h}{\delta}\widetilde{\mathcal{M}}f(x_0) + \epsilon.\end{aligned}$$

Then by the arbitrariness of ϵ, it follows that

$$\frac{1}{h}[\widetilde{\mathcal{M}}f(x_0) - \widetilde{\mathcal{M}}(f)(x_0 - h)] \le \frac{1}{\delta}\widetilde{\mathcal{M}}f(x_0), \ \forall \ h > 0.$$

Similarly, we have for all $x, y \in (a, b)$ with $x > y$

$$\widetilde{\mathcal{M}}f(x) - \widetilde{\mathcal{M}}f(y) \le \frac{x - y}{\delta}\ \widetilde{\mathcal{M}}f(x).$$

On the other hand, let $\delta' = \min\{\delta, \frac{b - x_0}{2}\}$. Together with the monotonic property of $\widetilde{\mathcal{M}}f$ on (a, b),

$$\widetilde{\mathcal{M}}f(x) - \widetilde{\mathcal{M}}f(x_0) \le \frac{x - x_0}{\delta}\ \widetilde{\mathcal{M}}f(x) \le \frac{x - x_0}{\delta}\ \widetilde{\mathcal{M}}f(x_0 + \delta'), \ \forall \ x \in (x_0, x_0 + \delta').$$

So $(\widetilde{\mathcal{M}}f)'(x_0) \ne \infty$, and thus $G = \emptyset$, $|\widetilde{\mathcal{M}}f(G)| = 0$.

Moreover, if one-sided limits $\lim\limits_{x \to a^+} \widetilde{\mathcal{M}}f(x)$, $\lim\limits_{x \to b^-} \widetilde{\mathcal{M}}f(x)$ exist, one can easily check that there exists $C > 0$ such that $\widetilde{\mathcal{M}}f(x) \le C$ for all $x \in (a, b)$. Then we conclude for all $x, y \in (a, b)$ with $x > y$

$$\widetilde{\mathcal{M}}f(x) - \widetilde{\mathcal{M}}f(y) \le \frac{x - y}{\delta}\ \widetilde{\mathcal{M}}f(x) \le \frac{x - y}{\delta}C.$$

For all $x, y \in (a, b)$ with $x < y$

$$\widetilde{\mathcal{M}}f(y) - \widetilde{\mathcal{M}}f(x) \le \frac{y - x}{\delta}\ \widetilde{\mathcal{M}}f(y) \le \frac{y - x}{\delta}C.$$

This completes the proof of Lemma 5.2.4.

□

Now we are ready to prove the main result.

Theorem 5.2.5. *If $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{\mathcal{M}}f$ is weakly differentiable on $\mathbb{R}$, and*

$$\|\big(\widetilde{\mathcal{M}}f\big)'\|_{L^1(\mathbb{R})} \leq \|f'\|_{L^1(\mathbb{R})}.$$

Moreover, $\widetilde{\mathcal{M}}f$ is absolutely continuous, and

$$\mathrm{Var}\big(\widetilde{\mathcal{M}}f\big) \leq \mathrm{Var}(f).$$

Proof. Note that if $f \in W^{1,1}(\mathbb{R})$, then f is absolutely continuous on $\mathbb{R}$ and vanishes at infinity. By Lemma 5.2.2, we have $\widetilde{\mathcal{M}}f(x) \geq |f(x)|$ for all $x \in \mathbb{R}$. Meanwhile, $\widetilde{\mathcal{M}}f$ is continuous and vanishes at infinity. Therefore, the set $A := \{x \in \mathbb{R} : \widetilde{\mathcal{M}}f(x) > |f(x)|\}$ is open. We can write A as a countable union of disjoint open intervals:

$$A = \bigcup_j I_j := \bigcup_j (\alpha_j, \beta_j).$$

where $\widetilde{\mathcal{M}}f(\alpha_j) = |f(\alpha_j)|$ and $\widetilde{\mathcal{M}}f(\beta_j) = |f(\beta_j)|$. Moreover, if $\alpha_j = -\infty$ or $\beta_j = +\infty$, then $f(\alpha_j) = \widetilde{\mathcal{M}}f(\alpha_j) = 0$ and $f(\beta_j) = \widetilde{\mathcal{M}}f(\beta_j) = 0$.

For each interval I_j, since the constant segments are not allowed, we can claim that I_j satisfies only one of the following conditions:

(i) $\widetilde{\mathcal{M}}f$ is strictly increasing on I_j;

(ii) $\widetilde{\mathcal{M}}f$ is strictly decreasing on I_j;

(iii) there exists $b_j \in I_j$ such that $\widetilde{\mathcal{M}}f$ is strictly decreasing on (α_j, b_j) and is strictly increasing on (b_j, β_j).

Otherwise, there exist $\alpha_j < c_1 < c_2 < c_3 < \beta_j$ such that

$$\widetilde{\mathcal{M}}f(c_1) < \widetilde{\mathcal{M}}f(c_2),\ \widetilde{\mathcal{M}}f(c_2) \geq \widetilde{\mathcal{M}}f(c_3),$$

or

$$\widetilde{\mathcal{M}}f(c_1) \leq \widetilde{\mathcal{M}}f(c_2),\ \widetilde{\mathcal{M}}(f)(c_2) > \widetilde{\mathcal{M}}f(c_3).$$

It follows from Weierstrass extreme value theorem and the continuity of $\widetilde{\mathcal{M}}f$ that there exists a local maximum of $\widetilde{\mathcal{M}}f$ in $[c_1, c_3]$ at least. By Theorem 4.2.3 we can get a contradiction.

Below we shall conclude that $\widetilde{\mathcal{M}}f$ is absolutely continuous on I_j and

$$\mathrm{Var}(\widetilde{\mathcal{M}}f; I_j) \leq \mathrm{Var}(f; I_j), \tag{5.2.7}$$

where $\mathrm{Var}(f; I_j)$ denotes the total variation of f on I_j.

If I_j satisfies (i) or (ii), $\widetilde{\mathcal{M}}f$ is absolutely continuous on I_j which follows directly from Lemma 5.2.4. Obviously,

$$\mathrm{Var}(\widetilde{\mathcal{M}}f; I_j) = |\widetilde{\mathcal{M}}f(\beta_j) - \widetilde{\mathcal{M}}f(\alpha_j)| = ||f(\beta_j)| - |f(\alpha_j)|| \leq \mathrm{Var}(|f|; I_j) \leq \mathrm{Var}(f; I_j).$$

If I_j satisfies (iii),

$$\begin{aligned}\mathrm{Var}(\widetilde{\mathcal{M}}f; I_j) &= (\widetilde{\mathcal{M}}f(\beta_j) - \widetilde{\mathcal{M}}f(b_j)) + (\widetilde{\mathcal{M}}f(\alpha_j) - \widetilde{\mathcal{M}}f(b_j)) \\ &< (|f(\beta_j)| - |f(b_j)|) + (|f(\alpha_j)| - |f(b_j)|) \\ &\le |f(\beta_j) - f(b_j)| + |f(\alpha_j) - f(b_j)| \le \mathrm{Var}(f; I_j),\end{aligned}$$

which gives (5.2.7).

On the other hand, by Lemma 5.2.4 again we have $\widetilde{\mathcal{M}}f$ is absolutely continuous on (α_j, b_j), and on (b_j, β_j) respectively. Then $\widetilde{\mathcal{M}}f$ maps measure zero sets of (α_j, b_j) into measure zero sets, and maps measure zero sets of (b_j, β_j) into measure zero sets as well. For any measure zero set $G \subset I_j$, it is easy to see that

$$|\widetilde{\mathcal{M}}f(G)| \le |\widetilde{\mathcal{M}}f(G \cap (\alpha_j, b_j))| + |\widetilde{\mathcal{M}}f(G \cap (b_j, \beta_j))| = 0, \tag{5.2.8}$$

which shows that $\widetilde{\mathcal{M}}f$ maps measure zero sets of I_j into measure zero sets. This together with the fact that $\widetilde{\mathcal{M}}f$ is continuous and of bounded variation on I_j yields that $\widetilde{\mathcal{M}}f$ is absolutely continuous on I_j and thus has a weak derivative v on each I_j. Moreover, the weak derivative coincides with the classical derivative almost everywhere.

Next, similarly to Tanaka's arguments in [46], we will prove that $\widetilde{\mathcal{M}}f$ is weakly differentiable on $\mathbb{R}$ with

$$(\widetilde{\mathcal{M}}f)' = |f|'\chi_{A^c} + v\chi_A, \tag{5.2.9}$$

where χ_A and χ_{A^c} denote the characteristic functions of the sets A and A^c.

Indeed, notice that for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\widetilde{\mathcal{M}}f\varphi$ is absolutely continuous on I_j. Then applying integration by parts gives

$$\int_{I_j} \widetilde{\mathcal{M}}(f)(x)\varphi'(x)dx = (|f(\beta_j)|\varphi(\beta_j) - |f(\alpha_j)|\varphi(\alpha_j)) - \int_{I_j} v(x)\varphi(x)dx. \tag{5.2.10}$$

It follows from (4.2.10) that

$$\begin{aligned}&\int_{\mathbb{R}} \widetilde{\mathcal{M}}f(x)\varphi'(x)dx \\ &= \int_A \widetilde{\mathcal{M}}f(x)\varphi'(x)dx + \int_{A^c} \widetilde{\mathcal{M}}f(x)\varphi'(x)dx \\ &= \sum_j (|f(\beta_j)|\varphi(\beta_j) - |f(\alpha_j)|\varphi(\alpha_j)) - \int_A v(x)\varphi(x)dx + \int_{A^c} |f(x)|\varphi'(x)dx.\end{aligned} \tag{5.2.11}$$

Also, $f\varphi$ is absolutely continuous on I_j. Using integration by parts again, we get

$$\begin{aligned}
&\sum_j (|f(\beta_j)|\varphi(\beta_j) - |f(\alpha_j)|\varphi(\alpha_j)) - \int_A v(x)\varphi(x)dx + \int_{A^c} |f(x)|\varphi'(x)dx \\
&= \int_A |f(x)|\varphi'(x)dx + \int_A |f|'(x)\varphi(x)dx - \int_A v(x)\varphi(x)dx + \int_{A^c} |f(x)|\varphi'(x)dx \\
&= \int_{\mathbb{R}} |f(x)|\varphi'(x)dx + \int_A |f|'(x)\varphi(x)dx - \int_A v(x)\varphi(x)dx.
\end{aligned} \tag{5.2.12}$$

Obviously, $|f|$ is weakly differentiable on $\mathbb{R}$ because $|f|$ is absolutely continuous on $\mathbb{R}$. So for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|\varphi'(x)dx = -\int_{\mathbb{R}} |f|'(x)\varphi(x)dx.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} |f(x)|\varphi'(x)dx + \int_A |f|'(x)\varphi(x)dx - \int_A v(x)\varphi(x)dx \\
&= -\int_{\mathbb{R}} (|f|'(x)\chi_{A^c}(x) + v(x)\chi_A(x))\varphi(x)dx
\end{aligned}$$

which combining with (5.2.11)-(5.2.12) implies (5.2.9).

It follows from and (5.2.9) that

$$\begin{aligned}
\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} &= \int_A |(\widetilde{\mathcal{M}}f)'(x)|dx + \int_{A^c} |(\widetilde{\mathcal{M}}f)'(x)|dx \\
&= \int_A |v(x)|dx + \int_{A^c} ||f|'(x)|dx \\
&= \sum_j \mathrm{Var}(\widetilde{\mathcal{M}}f; I_j) + \int_{A^c} ||f|'(x)|dx.
\end{aligned}$$

By $\mathrm{Var}(\widetilde{\mathcal{M}}f; I_j) \le \mathrm{Var}(f; I_j)$, then we have

$$\begin{aligned}
\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} &\le \sum_j \mathrm{Var}(|f|; I_j) + \int_{A^c} ||f|'(x)|dx \\
&= \sum_j \int_{I_j} ||f|'(x)|dx + \int_{A^c} ||f|'(x)|dx \\
&\le \int_{\mathbb{R}} ||f|'(x)|dx = \mathrm{Var}(|f|) \le \mathrm{Var}(f) = \|f'\|_{L^1(\mathbb{R})}.
\end{aligned}$$

More exactly, if f is continuous and of bounded variation on $\mathbb{R}$, then $\mathrm{Var}(|f|) = \mathrm{Var}(f)$. Finally, since $\widetilde{\mathcal{M}}f$ is weakly differentiable on $\mathbb{R}$ and $(\widetilde{\mathcal{M}}f)' \in L^1(\mathbb{R})$, thus

$\widetilde{\mathcal{M}}f$ is absolutely continuous on $\mathbb{R}$ and

$$\operatorname{Var}(\widetilde{\mathcal{M}}f) = \|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})} = \operatorname{Var}(f).$$

$\square$

Compared with Tanaka's work, Tanaka [46] showed $\widetilde{M_l}f$, $\widetilde{M_r}f$ are absolutely continuous on each I_j, where

$$\widetilde{M_l}f(x) := \sup_{s>0} \int_{x-s}^{x} |f(y)|dy, \quad \widetilde{M_r}f(x) := \sup_{t>0} \int_{x}^{x+t} |f(y)|dy,$$

and

$$\|(\widetilde{M_l}f)'\|_1 \le \|f'\|_1, \quad \|(\widetilde{M_r}f)'\|_1 \le \|f'\|_1.$$

These together with the fact

$$\widetilde{M_l}f(x) = \max\{\widetilde{M_l}f(x), \widetilde{M_r}f(x)\}$$

imply that $\|(\widetilde{M}f)'\|_1 \le 2\|f'\|_1$. In our work, by the identity of the local maximum we obtain that $\widetilde{M}f$ is absolutely continuous on each I_j, then deduce a improved result $\|(\widetilde{\mathcal{M}}f)'\|_1 \le \|f'\|_1$.

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